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Abstract. We compute the expectations of the squares of the electric and magnetic fields in the vacuum region outside a half–space filled with a uniform non–dispersive dielectric. This gives predictions for the Casimir–Polder force on an atom in the 'retarded' regime near a dielectric. We also find a positive energy density due to the electromagnetic field. This would lead, in the case of two parallel dielectric half–spaces, to a positive, separation–independent contribution to the energy density, besides the negative, separation–dependent Casimir energy. Rough estimates suggest that for a very wide range of cases, perhaps including all realizable ones, the total energy density between the half–spaces is positive.

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1. Introduction

In this paper, we investigate the quantum electromagnetic field in the vacuum region outside a half–space filled with a uniform, non–dispersive dielectric. We compute the expectations of the squares of the electric and magnetic fields in this region. We have two motivations for this.

First, the problem is natural in the study of quantum optics. Indeed, other workers have already investigated some aspects of this situation, for example, the effects of a nearby dielectric on atomic transition rates (see, e.g., Khosravi and Loudon 1991, 1992). Here we compute the expectations of the squares of the electric and magnetic fields, thus providing predictions of the Casimir–Polder force on an (electrically or magnetically) polarizable atom near the dielectric. These predictions test the ultraviolet renormalization of the the theory at a deeper level than do the transition–rate ones.

Our main motivation, however, comes from the hypothesized uses of negative energy densities to fuel exotic general–relativistic and thermodynamic effects. Serious workers have considered the possibility that negative energy densities might give rise to "worm holes," "warp drives" and "time machines." Such predictions depend on being able to generate persistent negative energy densities. At present, the only way that this might be achieved within reasonably well–understood physics is via Casimir–type effects. In the original Casimir (1948) effect, for example, the energy density due to the quantum electromagnetic field between two perfect parallel plane conductors is predicted to be negative. It should immediately be remarked that this negative energy density has never been directly observed.§ Still, it is this prediction which has generated an enormous amount of theoretical work, because the possible consequences are so spectacular.

We wanted to know what would happen to the prediction of negative energy densities if the plates were no longer idealized as perfect conductors. A realistic treatment of this would require a theory of the quantum electromagnetic field in inhomogeneous absorptive and dispersive media at finite temperature. Such theories are only now under development (see, e.g., Matloob *et al.* 1995), so it seems wise to consider as a first step the case of a non–absorptive, non–dispersive medium at zero temperature. Thus we shall consider the case of a half–space filled with a material of (frequency–independent) dielectric constant ϵ. While the case of a perfect conductor is formally the limit $\epsilon \uparrow \infty$ of this, an imperfect conductor is not well represented by such a model with ϵ finite. So we shall not be able to make any positive predictions about the behavior of real conductors.

Still, our results are strong enough to bear on the case of conductors. We shall find that, for dielectrics, finite–ϵ effects *cannot* be neglected, especially in computations of

§ Laboratory experiments measure the force between the plates, that is, th component $\hat{T}_{zz}$ of the stress–energy (Sparnaay 1957, 1958; Lamoreaux 1997, Bordag *et al.* 1998). The energy density is $\hat{T}_{tt}$. These two operators do not commute. There is a connection between them, in that the long–time average of the force is minus the gradient of the energy, but present experiments seem far from being able to measure $\hat{T}_{tt}$. This operator may as a matter of principle not be directly observable; see Helfer 1998.

the electromagnetic contribution to the energy density. In the case of two parallel half–spaces, these finite–ϵ corrections do not alter the attractive nature of the Casimir force, but may contribute a positive, separation–independent energy density which dominates the negative, separation–dependent, Casimir energy density. This strongly suggests that only after a careful treatment of the physics of real conductors will we know whether the perfect–conductor idealization is adequate for computing the energy density in such cases.

It is not easy to say accurately and briefly why a finite dielectric constant should modify the energy density to this degree, because the physics is non–local and depends on quantum interference. The presence of a polarizable medium in a region alters the field operators, by causing reflection and refraction of modes at the boundary. If the geometry is particularly simple (a plane interface) and the reflection sufficiently idealized (a perfect conductor), one has a great deal of cancellation. Small deviations from these idealizations can potentially lead to large effects. This is beacuse the energy density and the squares of the field strengths are defined by ultraviolet–divergent integrals (and must be renormalized).

To explain the situation more quantitativley, we first review some aspects of the Casimir effect, and then discuss the idealizations that have been made and how they might be expected to be modified in a more realistic treatment.

Between two perfect parallel plane conductors, one finds that the renormalized energy density is given by

$$\langle \hat{T}_{00} \rangle_{\rm ren} = \frac{1}{2} \langle \hat{E}^2 + \hat{B}^2 \rangle_{\rm ren} = -\frac{\pi^2 \hbar c}{720 l^4} \,, \tag{1}$$

where l is the distance between the plates. That this is independent of position can be shown on invariance grounds (and relies on the ideal, perfect–conductor boundary conditions). However, the electric and magnetic fields are *not* position–independent; one finds

$$\langle \hat{E}^2 \rangle_{\rm ren} = -\frac{\pi^2 \hbar c}{720 l^4} + \frac{\pi^2 \hbar c}{16 l^4} \frac{3 - 2\sin^2(\pi z/l)}{\sin^4(\pi z/l)} \tag{2}$$

$$\langle \hat{B}^2 \rangle_{\rm ren} = -\frac{\pi^2 \hbar c}{720 l^4} - \frac{\pi^2 \hbar c}{16 l^4} \frac{3 - 2\sin^2(\pi z/l)}{\sin^4(\pi z/l)} \tag{3}$$

at distance z from one plate. Near one plate, as $z \downarrow 0$, we find the asymptotic forms

$$\langle \hat{E}^2 \rangle_{\rm ren} \sim +\frac{3\hbar c}{16\pi^2 z^4} \tag{4}$$

$$\langle \hat{B}^2 \rangle_{\rm ren} \sim -\frac{3\hbar c}{16\pi^2 z^4} \,. \tag{5}$$

In other words, the renormalized expectations of $\hat{E}^2$ and $\hat{B}^2$ both diverge near a perfectly conducting plate, but there is a perfect cancellation between the divergent terms, leaving only a finite result.

Several commments on this are in order. First, the negative expectation of $\hat{B}^2$ occurs because it is a renormalized quantity, and means that the fluctuations of $\hat{B}$ are less than those of the Minkowski vacuum. Second, the divergences of $\langle \hat{E}^2 \rangle_{\rm ren}$ and

$\langle \hat{B}^2 \rangle_{\rm ren}$ as $z \downarrow 0$ are not expected to be physical, but rather arise from the idealized boundary conditions used. A real conductor would not be well approximated by a perfect conductor within atomic distances, and probably not within its plasma wavelength. Thus the expressions (2)–(5) are really only expected to be valid when one is sufficiently far from the conductor to neglect atomic structure and finite skin–depth.

Still, one is led to ask what would happen if the antisymmetry between the divergent parts of $\langle \hat{E}^2 \rangle_{\rm ren}$ and $\langle \hat{B}^2 \rangle_{\rm ren}$ could be disturbed. Could one produce energy densities much greater in magnitude than the Casimir expression (1)? A natural way to try to do this is to replace the perfect conductor by a dielectric, and this is what we have done here. Of course, our model is not expected to be accurate within atomic distances or even scales of the order of a skin–depth. Still, we shall be able to draw some interesting conclusions.

We are able to compute $\langle \hat{E}_z^2 \rangle_{\rm ren}$, $\langle \hat{E}_{\rm T}^2 \rangle_{\rm ren}$, $\langle \hat{B}_z^2 \rangle_{\rm ren}$ and $\langle \hat{B}_{\rm T}^2 \rangle_{\rm ren}$ explicitly, as functions of the distance z from the dielectric boundary and of the dielectric susceptibility χ. The expressions have the form

$$\eta \hbar c / z^4 \tag{6}$$

where the ηs are transcendental functions of χ. (See equations 28, 30, 32, 34.) We find in particular that the energy density in the vacuum half–space has the form $\eta_\rho \hbar c/z^4$, where η_ρ is a positive function of χ. This means that the total energy per unit surface area of the electromagnetic field on the vacuum side,

$$\int_0^\infty (\eta_\rho \hbar c / z^4)\, \mathrm{d}z\,, \tag{7}$$

is divergent. This is unphysical and again can be ascribed to the oversimplification of our model, where all modes, of whatever frequency, are equally affected by the dielectric. In a more realistic model, the dielectric's atomic structure would be taken into account. This would mean that at small distances (of the order of the skin depth probably and at the atomic scale certainly) the energy density would not be given by $\eta_\rho \hbar c/z^4$, but by some other, presumably finite, expression. Correspondingly, we ought really to think of our theory as an effective field theory valid only up to frequencies corresponding to wavelengths of order the skin depth or so.

In the next section, we outline the technical details of the computations. In Section 3, we summarize the asymptotic behaviors of the squares of the ηs for the squares of the fields, and present the graphs of these functions. Section 4 summarizes the behavior of the expectation of the stress tensor. Section 5 contains discussions of the significance of our results, and Section 6 recapitualiates the main conclusions.

2. The Computation

2.1. The Orthonormal Eigenmodes

The case of a half–space uniformly filled with a dielectric has been studied earlier, and we shall use the orthonormal eigenmodes as given by Carniglia and Mandel (1971).

We shall take the z axis to be normal to the interface, with z increasing in the vacuum region. We take advantage of the translational symmetries in time and in the $x_{\mathrm{T}} = (x, y)$ directions to resolve all modes by Fourier transforms in these variables, with Fourier transform variables ω and k_{T}. These Fourier transform variables thus retain their senses on both sides of the interface.

The dielectric constant is $\epsilon = 1 + \chi$. The wave number in the z–direction is k in the vacuum and $\tilde{k}$ in the dielectric. Thus we have

$$\tilde{k}^2 + k_{\mathrm{T}}^2 = \epsilon\omega^2 \quad \text{for} \quad z < 0 \quad \text{(dielectric)} \tag{8}$$

$$k^2 + k_{\mathrm{T}}^2 = \omega^2 \quad \text{for} \quad z > 0 \quad \text{(vacuum)}. \tag{9}$$

In what follows $\hat{k_{\mathrm{T}}}$ and $\hat{e}_z$ are the unit vectors in the k_{T} and z–directions. In later sections, hats will indicate field operators, too, but no confusion should arise.

We shall only need the modes on the vacuum side of the interface. The transverse electric component of the 'electric' field mode (where E is normal to the plane of incidence) incident from the left ($\tilde{k} > 0$), is

$$E^{\mathrm{E}}{}_{\tilde{k}k_{\mathrm{T}}} = (2\epsilon)^{-1/2}\frac{2\tilde{k}}{\tilde{k}+k}\mathrm{e}^{\mathrm{i}kz}\mathrm{e}^{\mathrm{i}k_{\mathrm{T}}\cdot x_{\mathrm{T}}}(\hat{k_{\mathrm{T}}} \times \hat{e}_z)\,. \tag{10}$$

The transverse magnetic component of the 'electric' field mode incident from the left ($\tilde{k} > 0$) is

$$E^{\mathrm{M}}{}_{\tilde{k}k_{\mathrm{T}}} = (\sqrt{2}\omega)^{-1}\frac{2\tilde{k}}{\tilde{k}+\epsilon k}\mathrm{e}^{\mathrm{i}kz}\mathrm{e}^{\mathrm{i}k_{\mathrm{T}}\cdot x_{\mathrm{T}}}\left(k_{\mathrm{T}}\hat{e}_z - k\hat{k_{\mathrm{T}}}\right)\,. \tag{11}$$

The transverse electric component of the 'electric' field mode incident from the right ($k > 0$) is

$$E^{\mathrm{E}}{}_{kk_{\mathrm{T}}} = \sqrt{2}^{-1}\left(\mathrm{e}^{-\mathrm{i}kz} + \frac{k-\tilde{k}}{k+\tilde{k}}\mathrm{e}^{\mathrm{i}kz}\right)\mathrm{e}^{\mathrm{i}k_{\mathrm{T}}\cdot x_{\mathrm{T}}}(\hat{k_{\mathrm{T}}} \times \hat{e}_z)\,. \tag{12}$$

The transverse magnetic component of the 'electric' field mode incident from the right ($k > 0$) is

$$E^{\mathrm{M}}{}_{kk_{\mathrm{T}}} = (\sqrt{2}\omega)^{-1}\left(\mathrm{e}^{-\mathrm{i}kz}\left(k_{\mathrm{T}}\hat{e}_z + k\hat{k_{\mathrm{T}}}\right) + \frac{\epsilon k-\tilde{k}}{\epsilon k+\tilde{k}}\mathrm{e}^{\mathrm{i}kz}\left(k_{\mathrm{T}}\hat{e}_z - k\hat{k_{\mathrm{T}}}\right)\right)\mathrm{e}^{\mathrm{i}k_{\mathrm{T}}\cdot x_{\mathrm{T}}}\,. \tag{13}$$

The transverse electric component of the 'magnetic' field mode incident from the left ($\tilde{k} > 0$) is

$$B^{\mathrm{E}}{}_{\tilde{k}k_{\mathrm{T}}} = (2\epsilon\omega^2)^{-1/2}\frac{2\tilde{k}}{\tilde{k}+k}\mathrm{e}^{\mathrm{i}kz}\mathrm{e}^{\mathrm{i}k_{\mathrm{T}}\cdot x_{\mathrm{T}}}\left(k\hat{k_{\mathrm{T}}} - k_{\mathrm{T}}\hat{e}_z\right)\,. \tag{14}$$

The transverse magnetic component of the 'magnetic' field mode incident from the left ($\tilde{k} > 0$) is

$$B^{\mathrm{M}}{}_{\tilde{k}k_{\mathrm{T}}} = 2^{-1/2}\frac{2\tilde{k}}{\tilde{k}+\epsilon k}\mathrm{e}^{\mathrm{i}kz}\mathrm{e}^{\mathrm{i}k_{\mathrm{T}}\cdot x_{\mathrm{T}}}(\hat{k_{\mathrm{T}}} \times \hat{e}_z)\,. \tag{15}$$

The transverse electric component of the 'magnetic' field mode incident from the right ($k > 0$) is

$$B^{\mathrm{E}}{}_{kk_{\mathrm{T}}} = -2^{-1/2}\omega^{-1}\left(\mathrm{e}^{-\mathrm{i}kz}\left(k\hat{k_{\mathrm{T}}} + k_{\mathrm{T}}\hat{e}_z\right) - \frac{k-\tilde{k}}{k+\tilde{k}}\mathrm{e}^{\mathrm{i}kz}\left(k\hat{k_{\mathrm{T}}} - k_{\mathrm{T}}\hat{e}_z\right)\right)\mathrm{e}^{\mathrm{i}k_{\mathrm{T}}\cdot x_{\mathrm{T}}}\,. \tag{16}$$

The transverse magnetic component of the 'magnetic' field mode incident from the right ($k > 0$) is

$$B^{\mathrm{M}}{}_{kk_{\mathrm{T}}} = 2^{-1/2}\left(\mathrm{e}^{-\mathrm{i}kz} + \frac{\epsilon k - \tilde{k}}{\tilde{k}+\epsilon k}\mathrm{e}^{\mathrm{i}kz}\right)\mathrm{e}^{\mathrm{i}k_{\mathrm{T}}\cdot x_{\mathrm{T}}}(\hat{k_{\mathrm{T}}}\times\hat{e}_z)\,. \tag{17}$$

The electric and magnetic field operators are thus given by

$$\begin{aligned}\hat{E}(\mathbf{x},t) = &\frac{1}{(2\pi)^3}\int_{\tilde{k}>0}\mathrm{d}^3\tilde{k}\sum_{\lambda=\mathrm{E,M}}\sqrt{\omega}\left(\hat{a}^{\lambda}_{\tilde{k}k_{\mathrm{T}}}E^{\lambda}_{\tilde{k}k_{\mathrm{T}}}\mathrm{e}^{-\mathrm{i}\omega t} + \mathrm{h.c.}\right)\\ &+\frac{1}{(2\pi)^3}\int_{k>0}\mathrm{d}^3k\sum_{\lambda=\mathrm{E,M}}\sqrt{\omega}\left(\hat{a}^{\lambda}_{kk_{\mathrm{T}}}E^{\lambda}_{kk_{\mathrm{T}}}\mathrm{e}^{-\mathrm{i}\omega t} + \mathrm{h.c.}\right)\end{aligned} \tag{18}$$

and

$$\begin{aligned}\hat{B}(\mathbf{x},t) = &\frac{1}{(2\pi)^3}\int_{\tilde{k}>0}\mathrm{d}^3\tilde{k}\sum_{\lambda=\mathrm{E,M}}\sqrt{\omega}\left(\hat{a}^{\lambda}_{\tilde{k}k_{\mathrm{T}}}B^{\lambda}_{\tilde{k}k_{\mathrm{T}}}\mathrm{e}^{-\mathrm{i}\omega t} + \mathrm{h.c.}\right)\\ &+\frac{1}{(2\pi)^3}\int_{k>0}\mathrm{d}^3k\sum_{\lambda=\mathrm{E,M}}\sqrt{\omega}\left(\hat{a}^{\lambda}_{kk_{\mathrm{T}}}B^{\lambda}_{kk_{\mathrm{T}}}\mathrm{e}^{-\mathrm{i}\omega t} + \mathrm{h.c.}\right)\,,\end{aligned} \tag{19}$$

where the creation and annihilation operators satisfy the commutation relations

$$\left[\hat{a}^{\lambda'}_{\tilde{k}'k_{\mathrm{T}}'}, \hat{a}^{\lambda *}_{\tilde{k}k_{\mathrm{T}}}\right] = 4\pi^3\hbar\delta_{\lambda\lambda'}\delta(\tilde{k}-\tilde{k}')\delta(k_{\mathrm{T}}-k_{\mathrm{T}}'), \tag{20}$$

and

$$\left[\hat{a}^{\lambda'}_{k'k_{\mathrm{T}}'}, \hat{a}^{\lambda *}_{kk_{\mathrm{T}}}\right] = 4\pi^3\hbar\delta_{\lambda\lambda'}\delta(k-k')\delta(k_{\mathrm{T}}-k_{\mathrm{T}}')\,. \tag{21}$$

2.2. Computation of $\hat{E}_z^2$

We shall outline the computation of $\hat{E}_z^2$. Computations of the squares of the other field components follow the same pattern.

We use a standard point–splitting in imaginary time, and set $\mathrm{i}\tau = t' - t$. Then we have

$$\begin{aligned}\langle\hat{E}_z^2\rangle = &\frac{1}{(2\pi)^3}\int_{\tilde{k}>0}\mathrm{d}^3\tilde{k}\frac{{k_{\mathrm{T}}}^2}{2\omega}\left(\frac{2\tilde{k}}{\tilde{k}+\epsilon k}\right)\left(\frac{2\tilde{k}}{\tilde{k}+\epsilon k}\right)^*\mathrm{e}^{\mathrm{i}(k-k^*)z}\mathrm{e}^{-\omega\tau}\\ &+\frac{1}{(2\pi)^3}\int_{k>0}\mathrm{d}^3k\frac{{k_{\mathrm{T}}}^2}{2\omega}\left(\mathrm{e}^{-\mathrm{i}kz} + \frac{\epsilon k-\tilde{k}}{\epsilon k+\tilde{k}}\mathrm{e}^{\mathrm{i}kz}\right)\left(\mathrm{e}^{-\mathrm{i}kz} + \frac{\epsilon k-\tilde{k}}{\epsilon k+\tilde{k}}\mathrm{e}^{\mathrm{i}kz}\right)^*\mathrm{e}^{-\omega\tau}\end{aligned} \tag{22}$$

We rewrite the integral over $\tilde{k} > 0$ as an integral over $k > 0$ (representing plane waves) plus an integral over $0 < \kappa < \omega\sqrt{\chi}$ (representing evanescent waves):

$$\langle\hat{E}_z^2\rangle = \frac{1}{(2\pi)^2}\int_0^{\infty}\mathrm{d}\omega\int_0^{\omega}\mathrm{d}k(\omega^2-k^2)\left(1 + \frac{\epsilon k-\tilde{k}}{\epsilon k+\tilde{k}}\cos 2kz\right)\mathrm{e}^{-\omega\tau}$$

$$+\frac{1}{(2\pi)^2}\int_0^\infty \mathrm{d}\omega \int_0^{\omega\sqrt{\chi}} \mathrm{d}\kappa(\omega^2+\kappa^2)\frac{2\epsilon\kappa\tilde{k}}{\tilde{k}^2+\epsilon^2\kappa^2}\mathrm{e}^{-2\kappa z}\mathrm{e}^{-\omega\tau}\,, \tag{23}$$

where we have used the relationships

$$k_{\mathrm{T}}{}^2 = \begin{cases} \omega^2+\kappa^2 & \text{for} \quad \tilde{k}<\omega\sqrt{\chi} \\ \omega^2-k^2 & \text{for} \quad \tilde{k}>\omega\sqrt{\chi} \end{cases} \tag{24}$$

and have performed the simple polar angle integration.

Changing variables $k=\omega\xi$ in the first integral and $\kappa=\omega\sqrt{\chi}\xi$ in the second integral we obtain

$$\begin{aligned}\langle \hat{E}_z^2\rangle = {} & \frac{1}{(2\pi)^2}\int_0^\infty \mathrm{d}\omega \int_0^1 \mathrm{d}\xi\,\omega^3(1-\xi^2)\left(1+\frac{\epsilon\xi-\sqrt{\chi+\xi^2}}{\epsilon\xi+\sqrt{\chi+\xi^2}}\cos 2\omega\xi z\right)\mathrm{e}^{-\omega\tau} \\ & +\frac{1}{(2\pi)^2}\int_0^\infty \mathrm{d}\omega \int_0^1 \mathrm{d}\xi\,\omega^3(1+\sqrt{\chi}\xi^2)\frac{2\epsilon\xi\sqrt{\chi+\xi^2}}{1+(\epsilon^2-1)\xi^2}\mathrm{e}^{-2\omega\sqrt{\chi}\xi z}\mathrm{e}^{-\omega\tau}\,. \end{aligned} \tag{25}$$

Integrating over ω gives

$$\begin{aligned}\langle \hat{E}_z^2\rangle = {} & \frac{\hbar c}{(2\pi)^2}\int_0^1 \left[6\frac{1-\xi^2}{\tau^4}+6(1-\xi^2)\frac{\epsilon\xi-\sqrt{\chi+\xi^2}}{\epsilon\xi+\sqrt{\chi+\xi^2}}\frac{16z^4\xi^4-24z^2\xi^2\tau^2+\tau^4}{(4z^2\xi^2+\tau^2)^4}\right. \\ & \left.+\frac{12\epsilon\sqrt{\chi}\xi(1+\chi\xi^2)\sqrt{1-\xi^2}}{(1+(\epsilon^2-1)\xi^2)(2z\sqrt{\chi}\xi+\tau)^4}\right]\mathrm{d}\xi\,. \end{aligned} \tag{26}$$

Thus we have reduced the problem of finding the expectation value of the square of the z–component of the electric field to a one–dimensional integral. The integral can be evaluated using contour integration in the complex plane and by exploiting Cauchy's residue theorem. After integrating and extensive algebra we obtain the following formally divergent expression for the expectation value:

$$\begin{aligned}\langle \hat{E}_z^2\rangle = \lim_{\tau\to 0}\Bigg(& \frac{\hbar c}{\pi^2\tau^4}+\frac{\hbar c}{(2\pi)^2 z^4}\left[\frac{1}{16\chi^{3/2}}\Bigg(2\sqrt{\chi}(6\epsilon^2-3\epsilon^{3/2}-2\chi)\right. \\ & +6\epsilon(1-2\epsilon^2+2\chi)\ln(\sqrt{\epsilon}+\sqrt{\chi}) \\ & \left.+\frac{6\epsilon^2(\epsilon^2-\chi-1)}{\sqrt{\epsilon^2-1}}\ln\left(\frac{\sqrt{\epsilon+1}-1}{\sqrt{\epsilon+1}+1}(\sqrt{\epsilon+1}+\sqrt{\epsilon})^2\right)\Bigg)\right]+O(\tau)\Bigg)\,. \end{aligned} \tag{27}$$

Subtracting the (again divergent) vacuum (Minkowski space) expectation value $\langle \hat{E}_z^2\rangle$Minkowski $=\lim_{\tau\to 0}\hbar c/\pi^2\tau^4$ and taking the limit as $\tau\to 0$ gives the exact renormalized expectation value:

$$\begin{aligned}\langle \hat{E}_z^2\rangle_{\mathrm{ren}} = {} & \frac{\hbar c}{(2\pi)^2 z^4}\left[\frac{1}{16\chi^{3/2}}\Bigg(2\sqrt{\chi}(6\epsilon^2-3\epsilon^{3/2}-2\chi)\right. \\ & +6\epsilon(1-2\epsilon^2+2\chi)\ln(\sqrt{\epsilon}+\sqrt{\chi}) \\ & \left.+\frac{6\epsilon^2(\epsilon^2-\chi-1)}{\sqrt{\epsilon^2-1}}\ln\left(\frac{\sqrt{\epsilon+1}-1}{\sqrt{\epsilon+1}+1}(\sqrt{\epsilon+1}+\sqrt{\epsilon})^2\right)\Bigg)\right] \end{aligned} \tag{28}$$

$$= \frac{\hbar c\eta_z^E}{z^4}\,, \tag{29}$$

say, where the coefficient η_z^E is a function of χ.

The renormalized expectations of the squares of the other components can be calculated by the same techniques. They are given by:

$$\langle \hat{E}_{\mathrm{T}}^2 \rangle_{\mathrm{ren}} = \frac{\hbar c}{(2\pi)^2 z^4} \left[\frac{1}{16\chi^{3/2}} \left(2\sqrt{\chi}(6 - 3\sqrt{\epsilon} - 2\chi) - 6(1 - 2\epsilon\chi) \right. \right.$$
$$\left. \left. \cdot \ln(\sqrt{\epsilon} + \sqrt{\chi}) - \frac{6\epsilon^2\chi}{\sqrt{\epsilon^2 - 1}} \ln \left(\frac{\sqrt{\epsilon + 1} - 1}{\sqrt{\epsilon + 1} + 1} (\sqrt{\epsilon + 1} + \sqrt{\epsilon})^2 \right) \right) \right] \tag{30}$$
$$= \frac{\hbar c \eta_{\mathrm{T}}^E}{z^4} \, ; \tag{31}$$

$$\langle \hat{B}_z^2 \rangle_{\mathrm{ren}} = \frac{\hbar c}{(2\pi)^2 z^4} \left[\frac{1}{16\chi^{3/2}} \left(2\sqrt{\chi}(12 - 9\sqrt{\epsilon} - 2\chi) \right. \right.$$
$$\left. \left. - 6(1 - 2\chi) \ln \left(\sqrt{\epsilon} + \sqrt{\chi} \right) \right) \right] \tag{32}$$
$$= \frac{\hbar c \eta_z^B}{z^4} \, ; \tag{33}$$

and

$$\langle \hat{B}_{\mathrm{T}}^2 \rangle_{\mathrm{ren}} = \frac{\hbar c}{(2\pi)^2 z^4} \left[\frac{1}{16\chi^{3/2}} \left(2\sqrt{\chi}(6 + 6\epsilon^2 - 2\chi - 3(\epsilon + 2)\sqrt{\epsilon}) \right. \right.$$
$$+ 6(\epsilon - 2\epsilon^3 + 2\chi) \ln(\sqrt{\epsilon} + \sqrt{\chi})$$
$$\left. \left. + 6\epsilon^2 \sqrt{\epsilon^2 - 1} \ln \left(\frac{\sqrt{\epsilon + 1} - 1}{\sqrt{\epsilon + 1} + 1} (\sqrt{\epsilon + 1} + \sqrt{\epsilon})^2 \right) \right) \right] \tag{34}$$
$$= \frac{\hbar c \eta_{\mathrm{T}}^B B}{z^4} \, . \tag{35}$$

These expressions are very complicated, and the characters of the functions $\eta(\chi)$ will be investigated in the next section. For the present, we remark that the sucessful renormalization provides a very strong check on the computations, since the term of order τ^{-4} must cancel perfectly against the term from Minkowski space, and the remaining potential poles in τ (of orders τ^{-3}, τ^{-2} and τ^{-1}) must vanish identically. Another check is provided by the vanishing of $\langle \hat{T}_{zz} \rangle_{\mathrm{ren}}$, as will be discussed in Section 4.

3. The Squares of the Fields

In the previous section, we found the expectations of the squares of the fields explicitly. In each case the result had the form $\hbar c \eta / z^4$, where z was the distance to the interface and η was a complicated transcendental function of the susceptibility χ. In this section, we present the graphs of the functions η, as well as their limiting behaviors for $\chi \downarrow 0$ and $\chi \uparrow \infty$.

The functions $\eta_{z,\mathrm{T}}^{E,B}$ (scaled to have a common limiting value) are presented in figure 1. They are in each case monotonic and approach a constant value asymptotically, but the approach is extremely slow.

For $\chi \downarrow 0$, we have

$$\langle \hat{E}_z^2 \rangle_{\mathrm{ren}} = \frac{\hbar c}{(2\pi)^2} \frac{9}{80 z^4} \chi + O(\chi^2) \, , \tag{36}$$

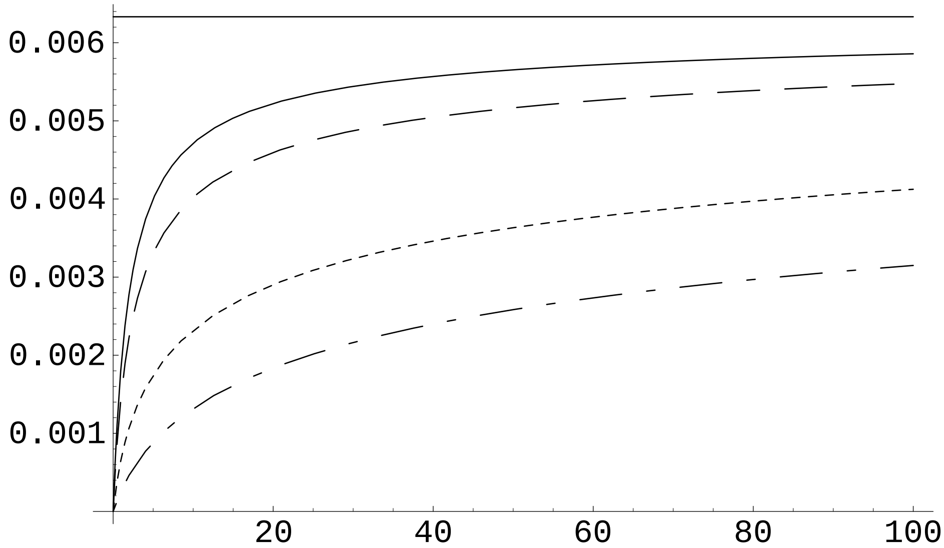

Figure 1. The dependences of the squares of the fields on the susceptibility χ (the abscissa). The horizontal line at the top is the common asymptote. Below that, in descending order, are η_z^E, then $(1/2)\eta_{\mathrm{T}}^E$, then $-(1/2)\eta_{\mathrm{T}}^B$, and finally $-\eta_z^B$.

$$\langle\hat{E}_{\mathrm{T}}^2\rangle_{\mathrm{ren}} = \frac{\hbar c}{(2\pi)^2}\frac{7}{40z^4}\chi + O(\chi^2) \tag{37}$$

$$\langle\hat{B}_z^2\rangle_{\mathrm{ren}} = -\frac{\hbar c}{(2\pi)^2}\frac{1}{80z^4}\chi + O(\chi^2)\,, \tag{38}$$

$$\langle\hat{B}_{\mathrm{T}}^2\rangle_{\mathrm{ren}} = -\frac{\hbar c}{(2\pi)^2}\frac{3}{40z^4}\chi + O(\chi^2)\,. \tag{39}$$

It is good to observe that all the above expectation values tend to zero as $\chi \downarrow 0$ (vacuum).

For $\epsilon \gg 1$ (that is, $\chi \uparrow \infty$), we find

$$\langle\hat{E}_z^2\rangle_{\mathrm{ren}} = \frac{\hbar c}{16\pi^2 z^4} - \frac{\hbar c}{(2\pi)^2}\frac{3}{16z^4}\frac{1}{\sqrt{\epsilon}} + O(1/\epsilon)\,, \tag{40}$$

$$\langle\hat{E}_{\mathrm{T}}^2\rangle_{\mathrm{ren}} = \frac{\hbar c}{8\pi^2 z^4} - \frac{\hbar c}{(2\pi)^2}\frac{3}{4z^4}\frac{1}{\sqrt{\epsilon}} + O(1/\epsilon)\,, \tag{41}$$

$$\langle\hat{B}_z^2\rangle_{\mathrm{ren}} = -\frac{\hbar c}{16\pi^2 z^4} + \frac{\hbar c}{(2\pi)^2}\frac{3}{8z^4}\frac{\ln\epsilon}{\sqrt{\epsilon}} + O(1/\sqrt{\epsilon})\,, \tag{42}$$

$$\langle\hat{B}_{\mathrm{T}}^2\rangle_{\mathrm{ren}} = -\frac{\hbar c}{8\pi^2 z^4} + \frac{\hbar c}{(2\pi)^2}\frac{3}{8z^4}\frac{\ln\epsilon}{\sqrt{\epsilon}} + O(1/\sqrt{\epsilon})\,. \tag{43}$$

We note that in the limit $\chi \uparrow \infty$, these quantities attain the values they would have in the case of the half–space outside a perfectly conducting plane (see *e.g.* Barton 1990). This is in accord with the usual formal identification of perfect conductors with dielectrics of infinite susceptibility. However, the approach to this limit is rather slow. One needs $\chi \approx 102$ for η_z^B to be within 50% of its limiting value, and $\chi \approx 14\,400$ to be within 10%.

For completeness, we list the limiting behaviors of squares of the full fields:

$$\langle \hat{E}^2 \rangle_{\rm ren} = \frac{\hbar c}{(2\pi)^2 z^4} \frac{23}{80} \chi + O(\chi^2) \tag{44}$$

$$\langle \hat{B}^2 \rangle_{\rm ren} = -\frac{\hbar c}{(2\pi)^2 z^4} \frac{7}{80} \chi + O(\chi^2) \tag{45}$$

for $\chi \downarrow 0$, and

$$\langle \hat{E}^2 \rangle_{\rm ren} = \frac{3\hbar c}{16\pi^2 z^4} - \frac{\hbar c}{(2\pi)^2 z^4} \frac{23}{80} \frac{1}{\sqrt{\epsilon}} + O(1/\epsilon) \tag{46}$$

$$\langle \hat{B}^2 \rangle_{\rm ren} = -\frac{3\hbar c}{16\pi^2 z^4} + \frac{\hbar c}{(2\pi)^2 z^4} \frac{3}{4} \frac{\ln \epsilon}{\sqrt{\epsilon}} + O(1/\sqrt{\epsilon}) \tag{47}$$

for $\chi \uparrow \infty$.

4. The Stress Tensor

By symmetry considerations, the expectation of the renormalized stress tensor must be diagonal. The 'z' component of the divergence constraint implies that $\langle \hat{T}_{zz} \rangle_{\rm ren}$ must be independent of z; however, as all components must be multiples of $1/z^4$, this component must be zero. (The verification that one does get zero using our values of the squares of the fields provides another check on our calculation.) Since the tensor is trace–free and the 'xx' and 'yy' components must be equal, there is only one algebraically independent component. We may take this to be the energy density $\rho = \langle \hat{T}_{tt} \rangle_{\rm ren}$, the other non–zero terms being $\langle \hat{T}_{xx} \rangle_{\rm ren} = \langle \hat{T}_{yy} \rangle_{\rm ren} = \rho/(2c^2)$.

The renormalized energy density is given by

$$\begin{aligned}
\langle \hat{T}_{00} \rangle_{\rm ren} = \frac{\hbar c}{(2\pi)^2 z^4} \Bigg[\frac{1}{16\chi^{3/2}} \Bigg(& 2\sqrt{\chi}(6\epsilon^2 + 12 - 4\chi - 3(\epsilon+3)\sqrt{\epsilon}) \\
& + 6(\epsilon - 1 - 2\epsilon^3 + 2(\epsilon+1)\chi) \ln(\sqrt{\epsilon} + \sqrt{\chi}) \\
& + \frac{6\epsilon^2(\epsilon^2 - \chi - 1)}{\sqrt{\epsilon^2 - 1}} \ln\left(\frac{\sqrt{\epsilon+1} - 1}{\sqrt{\epsilon+1} + 1} (\sqrt{\epsilon+1} + \sqrt{\epsilon})^2 \right) \Bigg) \Bigg]
\end{aligned} \tag{48}$$

$$= \frac{\hbar c \eta_\rho}{z^4}\,, \tag{49}$$

say.

The expectation of the energy density has the following limiting behaviours: for $\chi \ll 1$ we have

$$\langle \hat{T}_{00} \rangle_{\rm ren} = \frac{\hbar c}{(2\pi)^2 z^4} \left(\frac{\chi}{10} - \frac{2\chi^2}{35} \right) + O(\chi^3)\,, \tag{50}$$

and for $\epsilon \gg 1$ we have

$$\langle \hat{T}_{00} \rangle_{\rm ren} = \frac{\hbar c}{(2\pi)^2 z^4} \frac{3}{8} \frac{\ln \epsilon}{\sqrt{\epsilon}} + O(1/\sqrt{\epsilon})\,. \tag{51}$$

The graph of the coefficient η_ρ is given in figure 2. It has a single maximum with value $\eta_\rho \approx .0034$ at $\chi \approx 13.65$. The fall–off is again very slow, with η_ρ still at 50% its maximum value for $\chi \approx 600$, and still at 10% its maximum for $\chi \approx 60\,000$.

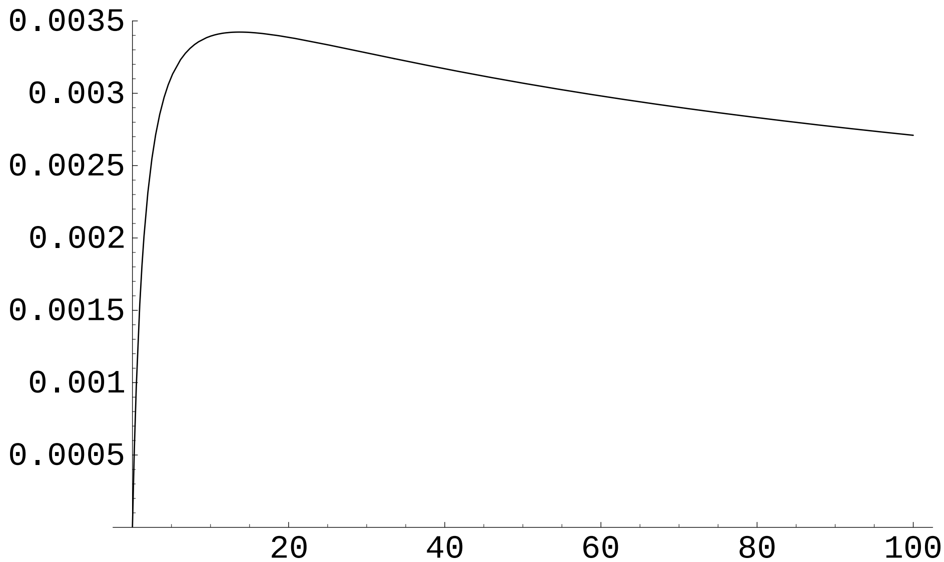

Figure 2. The dependence of the expectation of the energy density on the susceptibility. The coefficient η_ρ is plotted as a function of the susceptibility χ. There is a single maximimum at $\chi \approx 13.65$.

5. Discussion

We have found that the expectations of the renormalized squares of the components of the electric and magnetic fields at distance z from a non–dispersive dielectric all have the form $\eta\hbar c/z^4$, where the coeffcents η depend on the susceptibility χ. These coefficients vary linearly near $\chi = 0$, and tend monotonically but very slowly to asymptotic constant values as $\chi \uparrow \infty$. For the electric field components, one has $\eta > 0$; while for the magnetic field one has $\eta < 0$. For the energy density, the corresponding coefficient η_ρ rises linearly near $\chi = 0$, attains a maximum at $\chi \approx 13.65$, and then falls very slowly to zero as $\chi \uparrow \infty$.

Previous authors have considered the quantum optical effects of a non–dispersive dielectric half–space on atomic transition rates (e.g. Khosravi & Loudon 1991, 1992). Those results are in some sense complementary to these: those test the field operators over a small range of frequencies; whereas the present ones depend on integrating over all frequencies. The present results depend crucially on the successful ultraviolet renormalization of the theory.

5.1. Casimir–Polder Forces

An immediate consequence of these formulas is a prediction for the Casimir–Polder force on a polarizable atom near a dielectric. If the polarizability is $\alpha(\omega)$, then the induced dipole is $\int \alpha(\omega)\hat{E}(\omega)\,\mathrm{d}\omega$, and the potential energy is $-\left(\int \alpha(\omega)\hat{E}(\omega)\,\mathrm{d}\omega\right)\cdot\hat{E}$. If in the regime in question we may neglect the frequencey–dependence of the polarizability, then, in the vacuum state of the electromagnetic field the potential energy is the vacuum state of the electromagnetic field, this becomes

$$-\alpha\langle\hat{E}^2\rangle_{\rm ren} = -\alpha\hbar c\eta^E/z^4\,, \tag{52}$$

where $\eta^E = \eta^E_z + \eta^E_{\rm T}$. In principle, a parallel treatment applies to derive a 'magnetic Casimir–Polder' force depending on magnetic polarizability.

The Casimir–Polder force on a polarizable atom near a conductor has been measured in recent years (Sukenik *et al.* 1993), although as yet there has been no measurement near a dielectric.

5.2. Relation to Consitutive Energy

One of our main motivations for studying this model was to uncover its limitations. We noted early on that a real physical dielectric cannot be well approximated as a uniform medium on arbitrarily small scales. This means that we cannot expect our model to accurately capture the physics of the field modes whose wavelengths are less than the atomic scale (certainly) or the skin depth (probably). In particular, our predictions must break down as one gets within a distance of this order of the interface.

One vivid manifestation of this is the electromagnetic field energy of the vacuum half–space, per unit surface area. This surface energy density is

$$\int_0^\infty (\eta_\rho\hbar c/z^4)\,\mathrm{d}z\,, \tag{53}$$

which is *divergent* at the lower limit. (For an earlier investigation of a closely related effect, see Bordag and Lindig 1996.) It is not physically plausible that this energy density diverges; rather, the divergence reflects an improper model of physics near the interface. A more correct version would have the form

$$\text{local contribution near interface} + \int_\delta^\infty (\eta_\rho\hbar c/z^4)\,\mathrm{d}z\,, \tag{54}$$

where δ is of the order of the skin depth.

The energies we are considering here represent electromagnetic contributions to the constitutive energy of the medium. (In a more realistic treatment, it might not be meaningful to isolate one class of electromagnetic contributions from others, however.) It is plausible that these depend very much on the chemical physics of the material, and so the vagueness in the form (54) is apt. The numerical value of the contribution from the second term in (54) is quite modest for everyday materials. Taking the rather small value $\delta \approx 10\mathring{A}$ and $\eta_\rho = .003$, we find $\eta_\rho\hbar c/(3\delta^3) \sim 3\cdot 10^{-4}\mathrm{cal/cm}^2$.

5.3. The Fields in the Dielectric

In this paper, we have only treated the electromagnetic field in the vacuum half–space outside the dielectric. This case would seem to be of more interest than the field within the dielectric. Still, it is natural to ask what would happen there.

In principle, techniques like ours should apply to compute the operators $\hat{D} = \epsilon\hat{E}$ and $\hat{B}$ within the dielectric. The integrals involved are more difficult than those on the vacuum side, though.

Aside from technical difficulties in evaluating the integrals, there is another point which must be considered in the dielectric region. In that region, the ultraviolet asymptotics of the two–point functions are different than in Minkowski space. (Because the two–point functions are singular on characteristics, and there is a different speed of light in the dielectric medium.) This means that the renormalization cannot be accomplished by subtracting the Minkowski–space vacuum quantities. One could presumably renormalize by subtracting the quantities associated to a uniform dielectric. If one does this, then the local energy density differs from that of Minkowski space by an infinite amount.

We believe the resolution to this point is the same as that discussed in the previous subsection. One cannot accept the present model as an accurate picture of physics at all scales, and it is really only to be considered as an effective field theory, valid for frequencies below some cut–off. The difference in energy densities should be finite, with one contribution due to the effective field theory with a cut–off, and another due to the details of the chemical physics of the medium.

5.4. Sign of the Energy Density

One of our motivations for this work was to get a better understanding of the negative energy density occuring in the Casimir effect.

The Casimir effect — corresponding to two parallel plane conductors — is *formally* the limit as $\epsilon \uparrow \infty$ of two parallel dielectric half–spaces. However, this identification only holds at the limit, and only in the sense that in this case the reflections and refractions of the field modes at the dielectric interfaces approach the perfect–conductor boundary conditions as $\epsilon \uparrow \infty$. A real conductor, with finite conductivity, has a dielectric function which is significantly dispersive and absorptive, and cannot be modeled by a constant large real positive ϵ. Thus our present model cannot make any positive quantitative predictions about the Casimir effect for conductors.

However, we shall show that at least for dielectrics, the effects of a finite ϵ cannot be ignored, and that in realistic situations it seems most likely that the total expected energy density, including separation–independent contributions from the half–spaces, is *positive*. This suggests strongly that we must investigate the real physics of conductors before we can conclude that the total expected energy density between the plates is negative.

Consider two non–dispersive half–space dielectrics, of the same susceptibility, parallel and separated by a distance l. Then the total energy (per unit cross–sectional area) will have the form

$$E_{\mathrm{tot}} = E_\infty + \eta_2 \hbar c/l^3\,. \tag{55}$$

The function $\eta_2(\chi)$ has been computed by Lifshitz (1956). Here E_∞ is (twice) the energy of either dielectric in isolation.

For the energy density, let z with $-l/2 < z < l/2$ be a coordinate normal to the interface planes. Then near either interface one expects the energy density to be dominated by the physics of that interface, and so to be $\sim \eta_\rho \hbar c/(z \pm l/2)^4$, where $\eta_\rho(\chi)$ is the coefficient we computed previously, equation (48). We shall write these two contributions as $\rho_1(z)$ and $\rho_2(z)$. The total energy density will be

$$\rho_{\mathrm{tot}}(z) = \rho_1(z) + \rho_2(z) + \rho_{\mathrm{Cas}}(z,l)\,, \tag{56}$$

where $\rho_{\mathrm{Cas}}(z,l)$ must on dimensional grounds have the form $f(z/l)/l^4$, and be less singular at the interfaces than ρ_1, ρ_2. Indeed, the integral $\int_{-l/2}^{l/2} f(z/l)\mathrm{d}z = \eta_2 l/(\hbar c)$ must be finite. Thus f has at most mild singularities at the interfaces.

The form of the function f is at present unknown. As a very rough approximation, we shall assume it is constant in z/l, that is, the 'Casimir' contribution to the expected energy density is uniformly distributed between the half–spaces. We may ask if this value dominates the contribution $\rho_1 + \rho_2$ from the dielectrics, that is, if ρ_{tot} is positive or negative. This question can be answered by comparing our results with those of Lifshitz (1956), who computed the force of attraction of the two dielectrics. We find numerically that the energy density $\rho_1 + \rho_2$ dominates the average energy density unless $\chi \gtrsim 39\,000$. In other words, if the 'Casimir' contribution to the energy density is distributed uniformly, the total energy density would be everywhere positive between the half–spaces unless χ could be made to exceed $\approx 39\,000$.

A real dielectric exhibits absorption and dispersion; we can expect our model to be valid at distances of order z if the dielectric susceptibility is (nearly) a real positive constant for several orders of magnitude of frequency bracketing c/z. There seems to be nothing in the Kramers–Kronig relations preventing this from holding for the sorts of values of χ discussed above. Still, the scales are extreme enough that one wonders whether such susceptibility functions are more mathematical curiosities than physical possibilities. In other words, unless remarkable materials exist, with $\chi(\omega)$ approximately a real positive constant $\gtrsim 39\,000$ for several orders of magnitude of ω, it seems unlikely the total expected energy density anywhere between the dielectric half–spaces will be negative.

We should like to emphasize that while the contributions from the dielectrics probably make the energy density positive, their separation–independence ensures that they do not alter the usual predictions of the attractive force between the dielectrics. Indeed, we have used Lifshitz's results in our argument, and we accept his values for the force.

6. Conclusion

Negative energy densities were first discovered in quantum field theory with Casimir's prediction of an attractive force between two parallel perfect plane conductors. Since then, there has been considerable speculation on what the physical consequences of these negative energy densities might be.

The present model was introduced as a first step away from the idealization of boundary conditions induced by a dielectric of infinite susceptibility. It has been chosen for its relative mathematical simplicity, and it is unrealistic in that it neglects dispersion and absorption. Still, we find that the effects of finite–susceptibility contributions go as $\epsilon^{-1/2}\ln\epsilon$ and can be very significant: in the range that our model is likely to be valid, it seems that these contributions can make the *total* energy density between two dielectric half–spaces positive, while preserving the attractive force found by Lifshitz.

One cannot, from our model, draw any definite conclusion about the behavior of the energy density between two real conducting plates. What sorts of separation–independent corrections there are, due to finite conductivity, are at present unknown. But it does seem clear that we will only be justified in having confidence in theoretical predictions of the energy density between two real conducting plates if we take into account finite–conductivity effects.

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