1993

Black Holes and Singularities

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THE UNIVERSITY OF TULSA
THE GRADUATE SCHOOL

BLACK HOLES AND SINGULARITIES

by

Andrew Lang

A thesis submitted in partial fulfillment of
the requirements for the degree of Master of Science
in the Discipline of Applied Mathematics
The Graduate School
The University of Tulsa
1993

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THE GRADUATE SCHOOL

BLACK HOLES
AND
SINGULARITIES

A THESIS
APPROVED FOR THE DISCIPLINE OF
APPLIED MATHEMATICS

By Thesis Committe

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Abstract

Andrew Lang (Master of Science in Applied Mathematics)
Black Holes and Singularities (35 pp. - 4 Chapters)
Directed by Doctor Kevin O'Neil

(77 words)

First, I give definitions and mathematical preliminaries. Secondly, I give a history of the derivation of Einstein's field equations. From this basis, I present a derivation of Schwarzschild's solution. A discussion then follows of various types of black holes: stationary, charged, rotating, and charged/rotating. I also give a pictorial representation of the properties of each algebraically special solution. Finally, I present a general definition of singularities along with a discussion of closed trapped surfaces and naked singularities.

Acknowledgments

I would like to thank Dr. Kevin O'Neil for his guidance and support throughout my research. I would also like to thank Art Corcoran for the insight he gave me into the art of computing. Aditional thanks go to Adrian Kirk for his proof reading abilities. Finally, I would like to dedicate this thesis to my wife, Kelly Lang, for her patience and support.

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CHAPTER I

Einstein's Field Equations

"Absolute space, in its own nature, without relation to anything external, remains always similar and immovable."
"Absolute, true, and mathematical time, of itself, and from its own nature, flows equably without relation to anything external."

[FROM THE SCHOLIUM IN THE PRINCIPIA]

Sir Isaac Newton, 1642-1726

1.1 Metrics, Connections and Tensors

Definition

Given a vector space $\mathbf{T}$, a *metric tensor* $\boldsymbol{g}$ is a symmetric type $(0,2)$ tensor which is non-singular in the sense that $\boldsymbol{g}(\boldsymbol{\lambda}, \boldsymbol{\mu}) = 0, \ \forall \boldsymbol{\mu} \in \mathbf{T} \Rightarrow \boldsymbol{\lambda} = \mathbf{0}$.

Definition

A metric tensor $\boldsymbol{g}$ provides a vector space $\mathbf{T}$ with an *inner product* $\langle \boldsymbol{\lambda}, \boldsymbol{\mu} \rangle$ of vectors $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathbf{T}$ which is defined as follows: $\langle \boldsymbol{\lambda}, \boldsymbol{\mu} \rangle = \boldsymbol{g}(\boldsymbol{\lambda}, \boldsymbol{\mu}) = g_{ab} \lambda^a \mu^b, \ \forall \boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathbf{T}$.

Since the matrix $[g_{ab}]$ is non-singular it has an inverse $[g^{ab}]$ such that

$$g^{ab} g_{bc} = \delta^a_c. \tag{1.1}$$

It can be shown that g^{ab} are components of a type $(2,0)$ tensor $\hat{\mathbf{g}}$ [FN79]. In the same way as above, the contravariant metric tensor $\hat{\mathbf{g}}$ can be used to define an inner product on the vector space $\mathbf{T}^*$, the dual space of $\mathbf{T}$, as follows:

$$\langle \boldsymbol{\lambda}, \boldsymbol{\mu} \rangle = g^{ab} \lambda_a \mu_b, \ \forall \boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathbf{T}^*. \tag{1.2}$$

Definition

Given an N-dimensional differentiable manifold M, define the set of all tangent vectors at a point P as the *tangent space* of M at P, and denote it by $T_P(M)$.

Definition

Let P, Q be two neighbouring points with coordinates x^a and $x^a + \delta x^a$ respectively. If we have a one-to-one correspondance between $T_P(M)$ and $T_Q(M)$, we call the corresponding vectors *parallel.*

It is natural to require such a correspondance to be (a) linear, and (b) to reduce to the identity when $P = Q$. If we denote the vector at Q parallel to the vector λ^a at P by $\lambda^a + \delta^*\lambda^a$, then (a) implies

$$\lambda^a + \delta^*\lambda^a = Y_b^a \lambda^a, \tag{1.3}$$

where Y_b^a is a matrix depending only on P and Q. Whereas (b) is satisfied, to first order in δx^a, if

$$Y_b^a = \delta_b^a - \Gamma_{bc}^a \delta x^c, \tag{1.4}$$

where the Γ_{bc}^a depend only on P. We can therefore say that $\lambda^a + \delta^*\lambda^a$, the vector at Q coordinates $x^a + \delta x^a$, is parallel to λ^a the vector at P coordinates x^a, if

$$\lambda^a + \delta^*\lambda^a = (\delta_b^a - \Gamma_{bc}^a \delta x^c)\lambda^b, \tag{1.5}$$

$$\Rightarrow \delta^*\lambda^a = -\Gamma_{bc}^a \lambda^b \delta x^c. \tag{1.6}$$

The quantities Γ_{bc}^a are called *connection coefficients.*

Definition

The *absolute derivative* $\frac{D\lambda^a}{du}$ of a vector field $\lambda^a(u)$ along a curve γ is defined to be

$$\frac{D\lambda^a}{du} = \frac{d\lambda^a}{du} + \Gamma_{bc}^a \lambda^b \frac{dx^c}{du}. \tag{1.7}$$

Definition

The *covariant derivative* $\lambda^a_{;c}$ of a vector field λ^a is defined to be

$$\lambda^a_{;c} = \frac{\partial \lambda^a}{\partial x^c} + \Gamma_{bc}^a \lambda^b = \lambda^a_{,c} + \Gamma_{bc}^a \lambda^b. \tag{1.8}$$

Now suppose that our connection is symmetric, i.e. $\Gamma_{bc}^a = \Gamma_{cb}^a$, and is such that if λ^a and μ^a are any parallel vector fields along any curve, then the inner product

$g_{ab}\lambda^a\mu^b$ is constant along that curve. That is, for any curve parametrised by u,

$$\begin{aligned} \frac{D\lambda^a}{du} = \frac{D\mu^a}{du} &= 0 \\ \Rightarrow \frac{D}{du}\left(g_{ab}\lambda^a\mu^b\right) &= 0 \\ \Rightarrow \left(\frac{Dg_{ab}}{du}\right)\lambda^a\mu^b + g_{ab}\left(\frac{D\lambda^a}{du}\right)\mu^b + g_{ab}\lambda^a\left(\frac{D\mu^b}{du}\right) &= 0 \\ \Rightarrow \left(\frac{Dg_{ab}}{du}\right)\lambda^a\mu^b &= 0 \\ \Rightarrow g_{ab;c}\lambda^a\mu^b\frac{dx^c}{du} &= 0 \quad . \end{aligned}$$

Since this must hold for all λ^a, μ^b and $\frac{dx^c}{du}$, our requirement that the inner product $g_{ab}\lambda^a\mu^b$ of two parallel vector fields along a curve is constant along that curve reduces to $g_{ab;c} = 0$, or

$$g_{ab,c} = \Gamma^d_{ac}g_{db} + \Gamma^d_{bc}g_{ad}. \tag{1.9}$$

Relabelling, we have

$$g_{bc,a} = \Gamma^d_{ba}g_{dc} + \Gamma^d_{ca}g_{bd}, \tag{1.10}$$

and

$$g_{ca,b} = \Gamma^d_{cb}g_{da} + \Gamma^d_{ab}g_{cd}. \tag{1.11}$$

Taking (1.9) + (1.10) − (1.11) and using symmetry properties, we get

$$2\Gamma^d_{ca}g_{db} = g_{ab,c} + g_{bc,a} - g_{ca,b},$$

and contracting with $\frac{1}{2}g^{eb}$ we get

$$\Gamma^e_{ca} = \frac{1}{2}g^{eb}(g_{ba,c} + g_{cb,a} - g_{ca,b}). \tag{1.12}$$

Thus the condition above determines the connection coefficients in terms of g_{ab} and its derivatives. The statement that there exists a unique symmetric connection which preserves inner products under parallel transport is known as the *fundamental theorem of Riemannian geometry.* This connection is called the *metric connection.*

If we define

$$\Gamma_{abc} = \frac{1}{2}(g_{ac,b} + g_{ba,c} - g_{bc,a}), \tag{1.13}$$

then equation (1.12) gives

$$\Gamma^a_{bc} = g^{ad}\Gamma_{dbc}, \tag{1.14}$$

and

$$\Gamma_{abc} = g_{ad}\Gamma^d_{bc}. \tag{1.15}$$

The traditional names for Γ_{abc} and Γ^a_{bc} are *Christoffel symbols of the first and second kinds* respectively. Now we are ready to derive some fundamental tensors important in general relativity. From the definition of covariant differentiation for a vector field λ_a, we get

$$\lambda_{a;b} = \lambda_{a,b} - \Gamma^d_{ab}\lambda_d,$$

further covariant differentiation gives

$$\begin{aligned}\lambda_{a;bc} &= (\lambda_{a;b})_{,c} - \Gamma^e_{ac}\lambda_{e;b} - \Gamma^e_{bc}\lambda_{a;e} \\ &= \lambda_{a,bc} - \Gamma^d_{ab,c}\lambda_d - \Gamma^d_{ab}\lambda_{d,c} - \Gamma^e_{ac}(\lambda_{e,b} - \Gamma^d_{eb}\lambda_d) - \Gamma^e_{bc}(\lambda_{a,e} - \Gamma^d_{ae}\lambda_d).\end{aligned}$$

Interchanging indices and subtracting we get

$$\lambda_{a;bc} - \lambda_{a;cb} = R^d_{abc}\lambda_d, \tag{1.16}$$

where

$$R^d_{abc} = \Gamma^d_{ac,b} - \Gamma^d_{ab,c} + \Gamma^e_{ac}\Gamma^d_{eb} - \Gamma^e_{ab}\Gamma^d_{ec}. \tag{1.17}$$

R^d_{abc} is a type $(1,3)$ tensor which is known as the *curvature tensor.* Since the connection coefficients are determined by the metric tensor and its derivatives, so is the curvature tensor. Thus for a type $(0,1)$ tensor field, a necessary and sufficient condition for the commutability of covariant differentiation is that $R^a_{bcd} = 0$. It turns out that this is the necessary and sufficient condition for the commutability of covariant differentiation for tensor fields of all types. We can thus make the following interpretation.

Definition

A manifold is *flat* if at each point of it $R^a_{bcd} = 0$, otherwise it is *curved.*

The curvature tensor satisfies important relations, for instance it can be shown that R^a_{bcd} satisfies

$$R^a_{bcd} + R^a_{cdb} + R^a_{dbc} = 0. \tag{1.18}$$

This relation is known as the *cyclic identity.* We can also contract R^a_{bcd} to form a new tensor known as the *Ricci tensor,* and we denote its components by

$$R_{ab} = R^c_{abc}. \tag{1.19}$$

We can further contract the Ricci tensor to get what is known as the *curvature scalar,*

$$R = g^{ab} R_{ab} = R^a_a. \tag{1.20}$$

We are now ready to follow Einstein's *"period of unremitting labour"* to discover the field equations of general relativity.

1.2 Einstein's Field Equations

Galileo's law states that if two objects, of different mass, are dropped together from the same height they will strike the ground at the same moment. Einstein felt that there was something suspect in the way Newton's theory of gravitation accounted for this. As a consequence of Galileo's law and the attempt to incorporate gravity into his special theory of relativity, Einstein proposed his principle of equivalence. It states, in a freely falling (non-rotating) laboratory occupying a small region of spacetime, the laws of physics are the laws of special relativity.

Now let us follow Einstein in one of his many thought experiments that arose from his principle of equivalence. Imagine a ray of light sent across an accelerating laboratory in space, because of the laboratory's "upward" acceleration, the ray

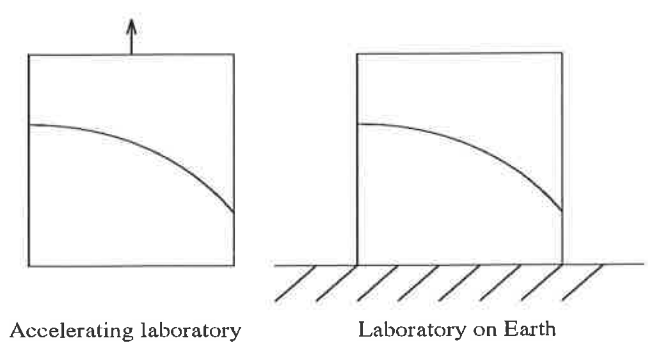

Figure 1.1: Einstein's Equivalent Laboratories

will seem to curve "downwards". Therefore, Einstein deduced, light sent in a ray across a laboratory on a gravitating Earth will also have to curve downwards. In other words, gravity bends light, see figure 1.1 above. For a full acount of Einstein's quest for the field equations see [Hof72] .

In 1911 he published a method of testing his hypothesis. He calculated that a ray of starlight grazing the Sun ought to be deflected by $0.83''$ [1] and later by his full fledged theory as $1.75''$, see table 1.1 for experimental support.

Table 1.1: The Angle of Deflection of Light Grazing the Sun

Date	Place of observation during a solar eclipse	Result
1919	Greenwich Observatory	$1.98 \pm 0.16''$
1922	Lick Observatory	$1.82 \pm 0.20''$
1947	Yerkes Observatory	$2.01 \pm 0.27''$
1972	Mullard Observatory	$1.82 \pm 0.14''$

The next revelation that came to Einstein was if all motion was to be relative, a variety of coordinate systems would apparently have to be tolerated, even if their relationship to direct measurement seemed next to impossible to specify. He realized that the equations of physics would have to be expressed in a way that would place all spacetime coordinate systems on an equal footing, a requirement that he later called *the principle of general covariance.* Einstein was in Prague at the time, and lacking the mathematical tools to apply this principle, he made little progress. In 1912 he returned to Zurich [2] and his mathematical helpmate Marcel Grossman. With the aid of Grossman, Einstein began to wield tensors, things he evidently had some difficulty in at first grasping, as expressed in a letter he wrote dated October 29^{th}, 1912 :

> "... I occupy myself exclusively with the problem of gravitation and now believe that I will overcome all difficulties with the help of a

[1] It should have been $0.87''$, but arithmetic was never one of Einstein's strong points.

[2] The place where ironically he had in 1895 failed his university entrance exam.

> friendly mathematician here. But this one thing is certain: that in all my life I have never before laboured at all as hard, and that I have become imbued with a great respect for mathematics, the subtle parts of which, in my innocence, I had till now regarded as pure luxury. Compared with this problem, the original theory of relativity is child's play."

In 1913 and again in 1914, he and Grossman published joint papers on their research. They believed that the metric tensor $g_{\mu\nu}$, which describes the geometry of spacetime, should depend on the amount of gravitating matter in the region in question. They therefore suggested the equation

$$g^{\mu\nu} = \kappa T^{\mu\nu}, \tag{1.21}$$

where κ is some coupling constant and $T^{\mu\nu}$ is the stress-energy tensor. $T^{\mu\nu}$ is a measure of the energy density, momentum density, and stress as measured by any and all observers at an event. Equation (1.21) looks plausible because $g^{\mu\nu}$ and $T^{\mu\nu}$ are symmetric, and $g^{\mu\nu}_{;\mu} = 0$ in agreement with $T^{\mu\nu}_{;\mu} = 0$. However equation (1.21) does not reduce to Poisson's equation,

$$\nabla^2 V = 4\pi G\rho, \tag{1.22}$$

in the Newtonian limit as it should do to be consistent. In retrospect it is amazing and heartbreaking how close they came to achieving their goal. Einstein later remarked that he and Grossman had considered the actual field equations only to discard them for what at the time seemed compelling reasons.

In 1915, Einstein published his hypothesis

$$R^{\mu\nu} = \kappa T^{\mu\nu}, \tag{1.23}$$

where $R^{\mu\nu}$ is the Ricci tensor, but $R^{\mu\nu}_{;\mu} \neq 0$. Later in the same year he modified the equation to what is known today as Einstein's field equations,

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu} = \kappa T^{\mu\nu}, \tag{1.24}$$

which reduces to Poisson's equation in the Newtonian limit and satisfies the condition $G^{\mu\nu}_{;\mu} = 0$.

The left hand side of equation (1.24) is the Einstein tensor. Comparing Einstein's equation with Poisson's, we identify the coupling constant κ as $\frac{-8\pi G}{c^4}$. Therfore an alternative form of Einstein's equations is

$$R^{\mu\nu} = \frac{-8\pi G}{c^4}(T^{\mu\nu} - \frac{1}{2}Tg^{\mu\nu}). \tag{1.25}$$

CHAPTER II

Schwarzschild's Solution

"A luminous star, of the same density as the Earth, and whose diameter should be two hundred and fifty times that of the Sun, would not, in consequence of its attraction, allow any of its light rays to arrive at us; it is therefore possible that the largest luminous bodies in the universe may, through this cause, be invisible."

P.S. Laplace(1798)

2.1 The Schwarzschild Solution

Einstein's equations are in general very difficult to solve as they possess a high degree of non-linearity. The problem becomes easier if we settle for algebraically special solutions. The first exact solution was obtained in just such a manner by Karl Schwarzschild in 1916. I will present here a simple derivation of his solution.

Assume that

(a) the field is spherically symmetric,

(b) the spacetime is empty, and

(c) the spacetime is asymptotically flat.

Schwarzschild postulated a line element of the form

$$c^2 d\tau^2 = A(r)dt^2 - B(r)dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\phi^2, \tag{2.1}$$

where $A(r)$ and $B(r)$ are to be determined. In flat spacetime the line element is

$$c^2 d\tau^2 = c^2 dt^2 - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\phi^2, \tag{2.2}$$

and thus assumption (c) imposes boundary conditions on $A(r)$ and $B(r)$ as follows:

$$A(r) \to c^2 \text{ and } B(r) \to 1 \text{ as } r \to \infty.$$

From equation (1.7), the condition for a *parallel field of vectors* along a curve in our spacetime is given by

$$\frac{d\lambda^a}{du} + \Gamma^a_{bc}\lambda^b \frac{dx^c}{du} = 0. \tag{2.3}$$

Definition

We may characterise a curve in a manifold as being a *geodesic* if there exists a parametrisation of it such that the tangent vectors $\lambda^a \equiv \frac{dx^a}{du}$ constitute a parallel vector field. Such a parameter is called an *affine parameter*. Substituting $\lambda^a = \frac{dx^a}{du}$ in equation (2.3) we get

$$\frac{d^2x^a}{du^2} + \Gamma^a_{bc}\frac{dx^b}{du}\frac{dx^c}{du} = 0, \tag{2.4}$$

as the equation for affinely parametrised geodesics in our spacetime manifold.

Now the Lagrangian of equation (2.1) is [FN79]

$$L(\dot{x}^\sigma, x^\sigma) = \frac{1}{2}(A(r)\dot{t}^2 - B(r)\dot{r}^2 - r^2\dot{\theta}^2 - r^2\sin^2\theta\,\dot{\phi}^2), \tag{2.5}$$

where dots denote differentiation with respect to an affine parameter u. Partial differentiation gives

$$\begin{aligned}
\frac{\partial L}{\partial \dot{t}} = A(r)\dot{t} \quad &\Rightarrow \quad \frac{d}{du}\left(\frac{\partial L}{\partial \dot{t}}\right) = A(r)\ddot{t} + A'(r)\dot{r}\dot{t}, \\
\frac{\partial L}{\partial \dot{r}} = -B(r)\dot{r} \quad &\Rightarrow \quad \frac{d}{du}\left(\frac{\partial L}{\partial \dot{r}}\right) = -B(r)\ddot{r} - B'(r)\dot{r}^2, \\
\frac{\partial L}{\partial \dot{\theta}} = -r^2\dot{\theta} \quad &\Rightarrow \quad \frac{d}{du}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = -r^2\ddot{\theta} - 2r\dot{r}\dot{\theta}, \\
\frac{\partial L}{\partial \dot{\phi}} = -r^2\sin^2\theta\,\dot{\phi} \quad &\Rightarrow \quad \frac{d}{du}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \sin^2\theta(-r^2\ddot{\phi} - 2r\dot{r}\dot{\phi} - 2r^2\cot\theta\,\dot{\theta}\dot{\phi}),
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial L}{\partial t} &= 0, \\
\frac{\partial L}{\partial r} &= \frac{1}{2}A'(r)\dot{t}^2 - \frac{1}{2}B'(r)\dot{r}^2 - r\dot{\theta}^2 - r\sin^2\theta\,\dot{\phi}^2, \\
\frac{\partial L}{\partial \theta} &= -r^2\sin\theta\cos\theta\,\dot{\phi}^2, \\
\frac{\partial L}{\partial \phi} &= 0,
\end{aligned}$$

where primes denote differentiation with respect to the radial coordinate r. Substitution of the above partial derivatives into the Euler-Lagrange equation

$$\frac{d}{du}\left(\frac{\partial L}{\partial \dot{x}^c}\right) - \frac{\partial L}{\partial x^c} = 0, \tag{2.6}$$

gives,

$$\ddot{t} + \frac{A'(r)}{A(r)}\dot{r}\dot{t} = 0, \tag{2.7}$$

$$\ddot{r} + \frac{A'(r)}{2B(r)}\dot{t}^2 + \frac{B'(r)}{2B(r)}\dot{r}^2 - \frac{r}{B(r)}\dot{\theta}^2 - \frac{r}{B(r)}\sin^2\theta\,\dot{\phi}^2 = 0, \tag{2.8}$$

$$\ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} - \sin\theta\cos\theta\,\dot{\phi}^2 = 0, \tag{2.9}$$

$$\ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} + 2\cot\theta\,\dot{\theta}\dot{\phi} = 0. \tag{2.10}$$

Now label the coordinates according to $x^0 \equiv t, x^1 \equiv r, x^2 \equiv \theta, x^3 \equiv \phi$, and write out the non-vanishing terms of equation (1.29) remembering to double the multipliers of the cross-terms $\dot{x}^\mu \dot{x}^\nu$ $(\mu \neq \nu)$ because they have two terms of the sum $\Gamma^\sigma_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$:

$$\ddot{t} + 2\Gamma^0_{01}\dot{r}\dot{t} = 0, \tag{2.11}$$

$$\ddot{r} + \Gamma^1_{00}\dot{t}^2 + \Gamma^1_{11}\dot{r}^2 + \Gamma^1_{22}\dot{\theta}^2 + \Gamma^1_{33}\dot{\phi}^2 = 0, \tag{2.12}$$

$$\ddot{\theta} + 2\Gamma^2_{12}\dot{r}\dot{\theta} + \Gamma^2_{33}\dot{\phi}^2 = 0, \tag{2.13}$$

$$\ddot{\phi} + 2\Gamma^3_{13}\dot{r}\dot{\theta} + 2\Gamma^3_{23}\dot{\theta}\dot{\phi} = 0. \tag{2.14}$$

Comparing equations (2.7)-(2.10) with equations (2.11)-(2.14), and equating, we get equations for the connection coefficients as follows:

$$\begin{aligned}
\Gamma^0_{01} &= \tfrac{A'(r)}{2A(r)}, & \Gamma^1_{00} &= \frac{A'(r)}{2B(r)},\\
\Gamma^1_{11} &= \tfrac{B'(r)}{2B(r)}, & \Gamma^1_{22} &= \frac{-r}{B(r)},\\
\Gamma^2_{12} &= \tfrac{1}{r}, & \Gamma^2_{33} &= -\sin\theta\cos\theta,\\
\Gamma^3_{13} &= \tfrac{1}{r}, & \Gamma^2_{23} &= \cot\theta.
\end{aligned}$$

From Equations (1.7) and (1.9) we have the Ricci tensor

$$R_{\mu\nu} = \Gamma^\sigma_{\mu\sigma,\nu} - \Gamma^\sigma_{\mu\nu,\sigma} + \Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\rho\nu} - \Gamma^\rho_{\mu\nu}\Gamma^\sigma_{\rho\sigma}.$$

By substituting the connection coefficients obtained above into $R_{\mu\nu} = 0$, we are using our results as a trial solution to the empty spacetime field equations and this gives

$$R_{00} = -\frac{A''(r)}{2B(r)} + \frac{A'(r)}{4B(r)}\left(\frac{A'(r)}{A(r)} + \frac{B'(r)}{B(r)}\right) - \frac{A'(r)}{rB(r)} = 0, \tag{2.15}$$

$$R_{11} = \frac{A''}{2A(r)} - \frac{A'(r)}{4A(r)}\left(\frac{A'(r)}{A(r)} + \frac{B'(r)}{B(r)}\right) - \frac{B'(r)}{rB(r)} = 0, \tag{2.16}$$

$$R_{22} = \frac{1}{B(r)} - 1 + \frac{r}{2B(r)}\left(\frac{A'(r)}{A(r)} - \frac{B'(r)}{B(r)}\right) = 0, \tag{2.17}$$

$$R_{33} = R_{22}\sin^2\theta = 0, \tag{2.18}$$

and $R_{\mu\nu} = 0$ identically when $\mu \neq \nu$. Adding $\frac{B(r)}{A(r)}$ times equation (2.15) to equation (2.16) gives

$$\frac{A'(r)}{A(r)} + \frac{B'(r)}{B(r)} = A'(r)B(r) + A(r)B'(r) = \frac{d}{dr}\left(A(r)B(r)\right) = 0. \tag{2.19}$$

This implies $A(r)B(r) =$ constant which can be identified using the boundary conditions as c^2. Substituting $B(r) = \frac{c^2}{A(r)}$ into equation (2.17) we get

$$\begin{aligned} A(r) + rA'(r) &= \frac{d}{dr}\left(rA(r)\right) = c^2, \\ \Rightarrow rA(r) &= c^2(r+k), \\ \Rightarrow \quad A(r) &= c^2\left(1 + \frac{k}{r}\right) \text{ and } B(r) = \left(1 + \frac{k}{r}\right)^{-1}. \end{aligned}$$

By comparing the Schwarzschild line element, in the asymptotic region, to the line element of flat spacetime in spherical polar coordinates we can conclude that [FN79]

$$k = -\frac{2MG}{c^2 r}. \tag{2.20}$$

Thus Schwarzschild's solution for the empty spacetime outside a spherical body of mass M is given by

$$c^2 d\tau^2 = c^2\left(1 - \frac{2MG}{c^2 r}\right)dt^2 - \left(1 - \frac{2MG}{c^2 r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2\sin^2\theta\, d\phi^2. \tag{2.21}$$

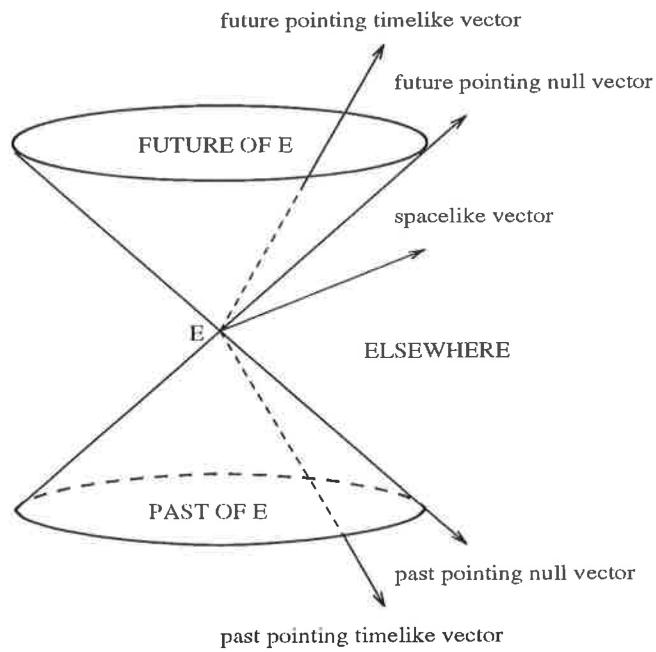

Figure 2.1: The Light Cone at an Event E

2.2 The Schwarzschild Black Hole

An important concept in relativity is that of a *light cone.* It is of central significance in the analysis of causality, and gives valuable insight concerning spacetime diagrams. An illustration of the definition of a light cone is in figure 2.1 above.

To be consistent with the standard in most texts, let us for convenience put $m = \frac{GM}{c^2}$ into Schwarzschild's line element which then becomes

$$c^2 d\tau^2 = \left(1 - \frac{2m}{r}\right) c^2 dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \; d\phi^2. \qquad (2.22)$$

Now suppose our massive object was a star that underwent symmetric gravitational collapse, concentrating all of its mass into a single point at $r = 0$. Then our line element becomes valid for all r with apparent *spacetime singularities* at $r = 0$ and $r = 2m$, where the metric components $g_{00} = (1 - \frac{2m}{r})$ and $g_{11} = (1 - \frac{2m}{r})^{-1}$ become infinite.

To help us understand exactly what is going on, let us examine the paths of the ingoing and outgoing radial null geodesics. Without loss of generality we can choose $\theta = \frac{\pi}{2}, \phi = 0$. This can be done because of the spherical symmetry of our

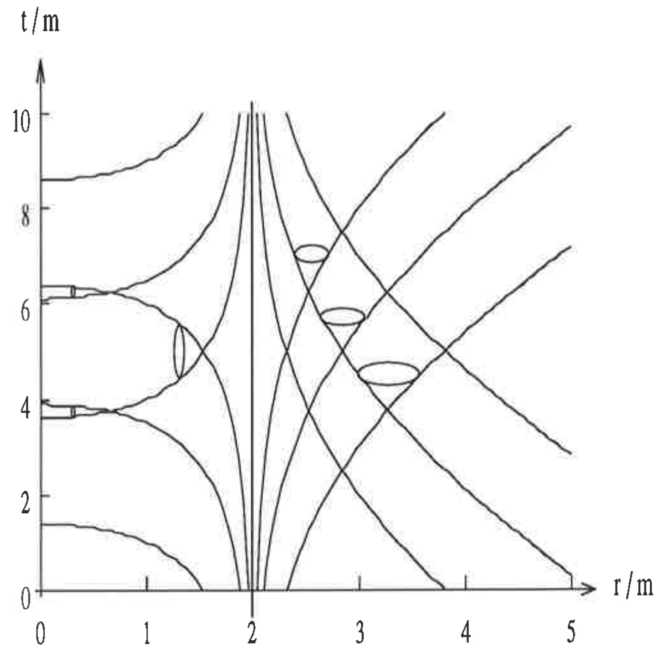

Figure 2.2: Radial Null Geodesics in Schwarzschild Coordinates

solution. From equation (2.22) we see that the radial null geodesics ($d\tau = 0$) are given by

$$\begin{aligned} 0 &= \left(1 - \frac{2m}{r}\right) c^2 dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2, \\ \Rightarrow \frac{dr}{dt} &= \pm \left(1 - \frac{2m}{r}\right) c, \\ \Rightarrow \quad t &= \pm \frac{1}{c} \left(r + 2m \ln |r - 2m| + B\right), \end{aligned} \tag{2.23}$$

where B is a constant of integration. We see that for the ingoing null geodesics $t \to \infty$ as $r \to 2m^+$. It therefore appears that light (or other particles of zero rest mass) aimed directly at our compacted star never gets past $r = 2m$. Does this mean the area between $r = 0$ and $r = 2m$ is unreachable by man, an area of the universe where we cannot go? The answer is no; the *Scwarzschild singularity* at $r = 2m$ is really not a spacetime singularity at all. It is just a pathology of the Schwazschild coordinates and given a suitable coordinate transformation, as we shall see later, we can transform it away. In general, singularities of this type, which can be transformed away, are known as *coordinate singularities.*

In the region $r > 2m$, the positive t direction is timelike ($g_{00} < 0$) and the inward r direction is spacelike ($g_{11} > 0$), as expected. On the contrary, for the region $r < 2m$, the positive t direction is spacelike ($g_{00} > 0$) and the inward r direction is timelike ($g_{11} < 0$). This can be seen easily by looking at the light cones in figure 2.2 above. What does it mean that the radial direction becomes timelike and the temporal direction becomes spacelike?

Imagine an astronaut in a spaceship travelling towards a collapsed star. He can (before he reaches $r = 2m$) always turn around if he wants to by using his rockets. The closer he gets to $r = 2m$, the harder it is for him to escape but in principle, given enough rocket power, he always can. Now if he decides to cross into the region $r < 2m$, a further decrease in his radial distance would correspond to a passing of time. This situation is interesting because with the passage of time his radial coordinate r **must** decrease; there is nothing the astronaut can do about it. No force can make time stand still, and thus his journey is fated to end with him crashing into the spacetime singularity at $r = 0$. Though he would most probably die before this due to the massive gravitational tidal forces acting on his body [MTW73] . Even light directed outward can never get past $r = 2m$ and the region, to an outside observer, appears completely black and is thus known commonly as a *black hole*. As a consequence of this, any events that occur within $r = 2m$ can never be communicated to the outside universe. Therefore we call the 4-sphere at $r = 2m$ the *event horizon* of the black hole. It seems that we can cross $r = 2m$, but once we have, we can **never** come back.

To get a line element that is valid for $r \leq 2m$ let us replace t by

$$v = ct + r + 2m \ln\left(\frac{r}{2m} - 1\right). \tag{2.24}$$

The coordinates (v, r, θ, ϕ) are known as *Eddington-Finkelstein coordinates* and the Schwarzschild line element in terms of them is

$$c^2 d\tau^2 = \left(1 - \frac{2m}{r}\right) dv^2 - 2dvdr - r^2 d\theta^2 - r^2 \sin^2\theta \; d\phi^2. \tag{2.25}$$

In our new coordinate system the radial null geodesics are given by

$$\left(1 - \frac{2m}{r}\right)\left(\frac{dv}{dr}\right)^2 - 2\frac{dv}{dr} = 0,$$

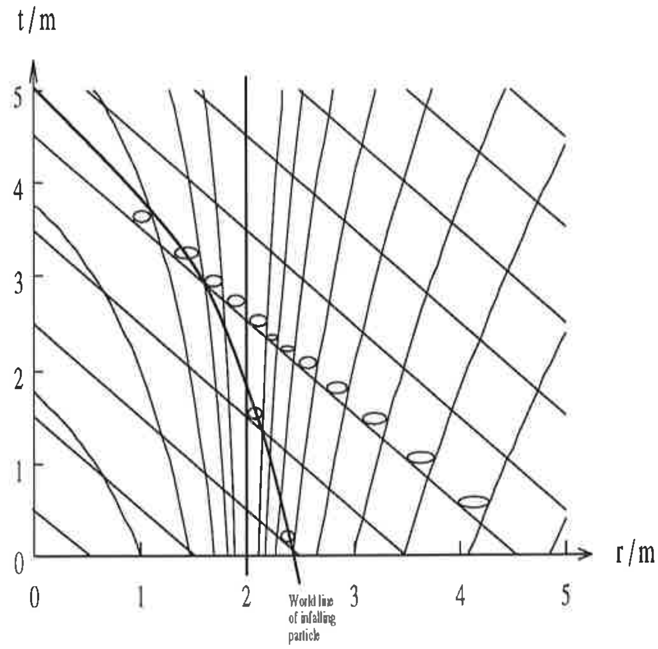

Figure 2.3: Radial Null Geodesics in Eddington-Finkelstein Coordinates

$$\Rightarrow \frac{dv}{dr} = 0 \quad \text{or} \quad \frac{dv}{dr} = \frac{2r}{r-2m},$$

and integrating we get

$$v = A \quad \text{or} \quad v = 2r + 4m \ln|r - 2m| + B, \tag{2.26}$$

where A and B are both constants.

In figure 2.3 above, oblique axes have been used so that the ingoing null geodesics are inclined at 45° just as they are in flat spacetime diagrams. We can see from the figure that the new line element is valid across $r = 2m$. Notice how the light cones tilt over as we approach the black hole, by the time we reach $r = 2m$, they have tilted so much that they actually point inwards towards the singularity at $r = 0$. Again this illustrates the fact that once inside $r = 2m$, nothing, not even light, can escape.

A superior coordinate system is given when we replace r and t by spacelike and timelike coordinates, u and v defined by

$$u = \left(1 - \frac{r}{2m}\right)^{\frac{1}{2}} e^{\frac{r}{4m}} \sinh\left(\frac{t}{4m}\right), \tag{2.27}$$

$$v = \left(1 - \frac{r}{2m}\right)^{\frac{1}{2}} e^{\frac{r}{4m}} \cosh\left(\frac{t}{4m}\right). \tag{2.28}$$

The coordinates (v, u, θ, ϕ) are known as *Kruskal-Szekeres coordinates.* The Schwazschild line element in terms of them is

$$c^2 d\tau^2 = 32\frac{m^3}{r} e^{-\frac{r}{2m}}(du^2 - dv^2) + r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{2.29}$$

where r is given implicitly by

$$\left(\frac{r}{2m} - 1\right) e^{\frac{r}{2m}} = u^2 - v^2. \tag{2.30}$$

Choosing $r = 0$, we get $u = \sinh\left(\frac{t}{4m}\right)$ and $v = \cosh\left(\frac{t}{4m}\right)$. The singularity at $r = 0$ is located in Kruskal-Szekeres coordinates at $v^2 - u^2 = 1$. Thus we have two singularities at $v = \pm(1 + u^2)^{\frac{1}{2}}$. Notice from equation (2.30) that the Schwarzschild region $r > 2m$ is now given by $u^2 > v^2$. Thus we not only have two singularities but also two exterior solutions $u > |v|$ and $u < -|v|$. This must mean that our original Schwarzschild coordinates and the Eddington-Finkelstein coordinates must be only a local coordinate patch on the full spactime manifold. By transforming from Schwarzschild coordinates to Kruskal-Szekeres coordinates, we have analytically extended the Schwarzschild metric to cover the whole manifold [MTW73] .

Figure 2.4 below shows that we need two separate Schwarzschild coordinate patches to cover the complete Schwarzschild geometry, whereas a single Kruskal-Szekeres coordinate system will suffice. The radial null geodesics are given by

$$32\frac{m^3}{r} e^{-\frac{r}{2m}}(du^2 - dv^2) = 0,$$

$$\Rightarrow \quad du = \pm dv,$$

$$\Rightarrow u = \pm v + A,$$

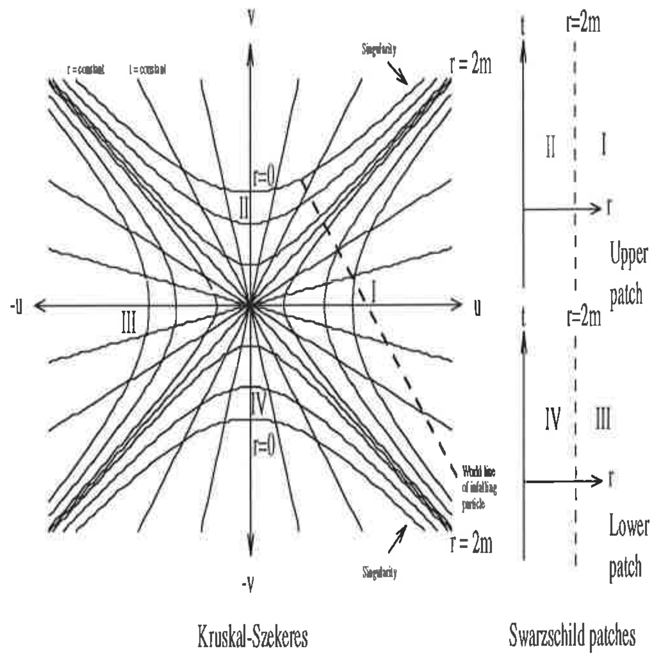

Figure 2.4: Radial Null Geodesics in Kruskal-Szekeres Coordinates

where A is a constant. In these coordinates the ingoing and outgoing null geodesics are naturally inclined at 45° just as they are in flat spacetime diagrams. This does not mean our spacetime is flat. Coordinate singularities, may be transformed away but no coordinate transformation can eliminate gravity or the physical spacetime singularity with infinite density, infinite spacetime curvature and infinite gravitational tidal forces at $r = 0$. These facts are summarized in the following theorem.

Birkhoff's theorem:

Let the geometry of a given region of spacetime

(1) be spherically symmetric, and

(2) be a solution to Einstein's field equations in vacuum.

Then that geometry is necessarily a piece of the Schwarzschild geometry.

For a proof of Birkhoff's theorem see [MTW73] .

The fact that the singularity at $r = 0$ really exists and cannot be avoided whatever coordinate system we choose, can be shown by calculating the *curvature invariant* :

$$I \equiv R_{abcd} R^{abcd} = 48 \frac{m^2}{r^6}.$$

Thus we can see that in every local Lorentz frame the spacetime curvature tends to infinity as $r \to 0$. We shall see later that the existence of singularities is closely related to the existence of closed trapped surfaces.

2.3 Motion in a Circle

From equation (2.5) we can write the Lagrangian in relativistic Schwarzschild coordinates ($c = 1$) in the equatorial plane ($\theta = \frac{\pi}{2}$) as

$$L = \frac{1}{2}\left[\left(1 - \frac{2m}{r}\right)\dot{t}^2 - \left(1 - \frac{2m}{r}\right)^{-1}\dot{r}^2 - r^2\dot{\phi}^2\right]. \tag{2.31}$$

Let dots denote differentiation with respect to τ. Partial differentiation of equation (2.31) gives the canonical momenta

$$p_t = \frac{\partial L}{\partial \dot{t}} = \left(1 - \frac{2m}{r}\right)\dot{t},$$
$$p_\phi = -\frac{\partial L}{\partial \dot{\phi}} = r^2\dot{\phi},$$

and

$$\frac{\partial L}{\partial t} = \frac{dp_t}{d\tau} = 0 \;\Rightarrow\; p_t = \left(1 - \frac{2m}{r}\right)\dot{t} = \text{constant}, \tag{2.32}$$
$$\frac{\partial L}{\partial \phi} = \frac{dp_\phi}{d\tau} = 0 \;\Rightarrow\; p_\phi = r^2\dot{\phi} = \text{constant}. \tag{2.33}$$

For null geodesics the Lagrangian must be equated to zero. Doing this we get

$$\left(1 - \frac{2m}{r}\right)^{-1}\left[p_t^2 - \left(\frac{dr}{d\tau}\right)^2\right] - \frac{p_\phi^2}{r^2} = 0,$$

and rearranging gives

$$\left(\frac{dr}{d\tau}\right)^2 + \frac{p_\phi^2}{r^2}\left(1 - \frac{2m}{r}\right) = p_t^2. \tag{2.34}$$

By considering $r = r(\phi)$ and replacing r by $u = \frac{1}{r}$ we get

$$f(u) = \left(\frac{du}{d\phi}\right)^2 = 2mu^3 - u^2 + \frac{1}{D^2}, \tag{2.35}$$

where $D = \frac{p_\phi}{p_t}$ denotes an *impact parameter*. Differetiation of equation (2.35) with respect to u and setting it equal to zero gives

$$f'(u) = 6mu^2 - 2u = 0,$$

which has a root $u = (3m)^{-1}$ or $r = 3m$. Now $u = (3m)^{-1}$ is a solution for $f(u) = 0$ if and only if $D^2 = 27m^2$. Thus for $D^2 = 27m^2$ and $u = (3m)^{-1}$, $\frac{du}{d\phi} = 0$ and thus an allowed null geodesic is a circular orbit of radius $3m$. Though it can be shown that this orbit is unstable [Cha83] .

Light orbiting around a black hole can produce startling effects. For instance, imagine a circular space-station around a black hole with radius $r = 3m$. A scientist on the space-station knows that it is curved, as he would have seen its shape on his journey to it. But looking down the corridor it appears completely straight, see figure 2.5 below. In fact, he would see the back of his head some distance down the corridor. Now suppose for a moment that he forgets that he

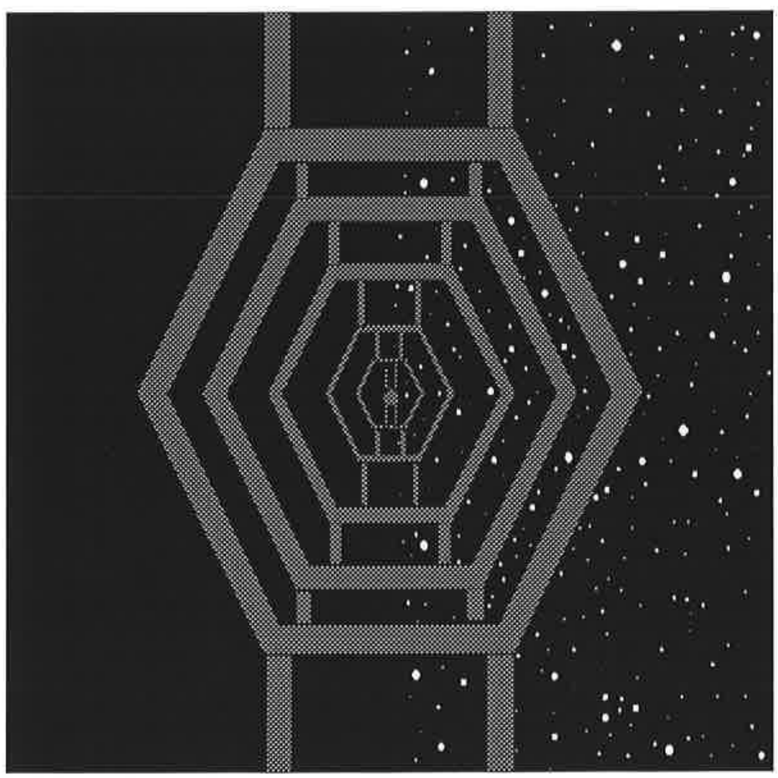

Figure 2.5: A Space-station at $r = 3m$

is on a circular space station. Thus he assumes that moving along the corridor, at any speed, he should not feel any centrifugal force. The only force he should feel is the one due to gravity. This seems absurd because he is really moving around in circles, not in straight lines. The faster he moves the more centrifugal force he should feel. Suprisingly though, whatever speed he goes he will not feel any centrifugal force, see [Abr93] . Thus not only does the space-station appear straight, it actually in a way, “acts” straight.

CHAPTER III

Other Solutions

"As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer ro reality."

[FROM GEOMETRY AND EXPERIENCE]

Albert Einstein, 1879-1955

3.1 The Reissner-Nordstrom Solution

The Reissner-Nordstom solution is an exact solution of Einstein's field equations. It describes a static spherically symmetric asymptotically flat spacetime outside a spherically symmetric charged massive body. We do not however expect any large object to have a significant net charge and thus the solution may not be a realistic one for black holes. The solution although algebraically special, as are all the solutions in this text, is still valid and examination of it can give us valuable insight into the nature of spacetime. It also provides a useful stepping stone to the Kerr solution given in the next section.

The line element of Reissner-Nordstrom spacetime is most commonly given in the form

$$c^2 d\tau^2 = \left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right) c^2 dt^2 - \left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\phi^2,$$

where m is the gravitational mass and e is the electric charge of the body. Suppose that our massive body was a star that underwent symmetric gravitational collapse, concentrating all of its mass and charge into a single point at $r = 0$. Then the solution is valid for all r with apparent singularities at $r = 0$ and $1 - \frac{2m}{r} + \frac{e^2}{r^2} = 0$.

The solution of $1 - \frac{2m}{r} + \frac{e^2}{r^2} = 0$ is given by

$$r_\pm = m \pm \sqrt{m^2 - e^2}, \tag{3.1}$$

and thus we have to consider two different situations:

(i) If $e^2 > m^2$ the metric is nonsingular everywhere except for the irremovable singularity at $r = 0$. In this case we have no event horizon to clothe the singularity at $r = 0$, and it is in plain view for the whole universe to see. A singularity which we can view is called a *naked singularity.* I will leave the discussion of naked singularities until later.

(ii) If $e^2 \leq m^2$ the metric has additional singularities at r_+ and r_-. Just as in the Schwarzschild metric, these singularities are coordinate singularities and given a suitable coordinate transformation we can transform them away.

For the following discussion assume $e^2 < m^2$, which will give us real and distinct r_+ and r_-. We shall see later that $r = r_+$ is an event horizon for the Reissner-Nordstrom black hole in the same sense that $r = 2m$ is an event horizon for the Schwarzschild black hole. It will also be seen that $r = r_-$ corresponds to a "horizon" of sorts.

To get a line element that is valid for all r not equal to zero, let us replace t by

$$v = ct + r + \frac{r_+^2}{r_+ - r_-}\ln(r - r_+) - \frac{r_-^2}{r_+ - r_-}\ln(r - r_-). \tag{3.2}$$

Our line element then becomes

$$c^2 d\tau^2 = \left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right) dv^2 - 2dvdr - r^2 d\theta^2 - r^2 \sin^2\theta \, d\phi^2. \tag{3.3}$$

In these coordinates the radial null geodesics are given by

$$\left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right)\left(\frac{dv}{dr}\right) - 2\frac{dv}{dr} = 0,$$

$$\Rightarrow \frac{dv}{dr} = 0 \quad \text{or} \quad \frac{dv}{dr} = \frac{2r^2}{r^2 - 2mr + e^2},$$

and integrating we get

$$v = A \quad \text{or} \quad v = 2r + \frac{2r_+^2}{r_+ - r_-}\ln|r - r_+| - \frac{2r_-^2}{r_+ - r_-}\ln|r - r_-| + B, \tag{3.4}$$

where A and B are both constants.

In figure 3.1 below the ingoing null geodesics are inclined at 45° just as

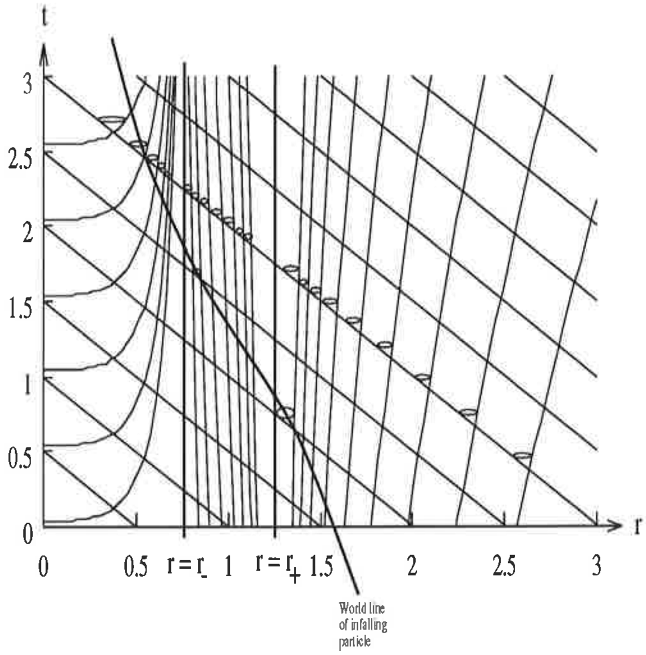

Figure 3.1: Radial Null Geodesics for the Reissner-Nordstom Solution

previously done in figure 2.3. Figure 3.1 shows that the line element is valid across both $r = r_+$ and $r = r_-$. Notice how the light cones tip over as we approach the black hole and by the time we reach $r = r_+$ they have tilted so far over that even "outward" directed light goes "inward". This confirms that $r = r_+$ is the event horizon for the Reissner-Nordstrom black hole.

Between r_+ and r_- the light cones point toward $r = r_-$ and all matter is forced inwards. But when we reach $r = r_-$, the light cones tip back towards vertical and the paths of particles curve back away from the singularity. The horizon $r = r_-$ is known commonly as a *Cauchy horizon.* The singularity appears repulsive; no timelike geodesic hits it, though non-geodesic timelike curves and radial null geodesics can.

If a falling observer is forced away from $r = 0$, he cannot escape past $r = r_+$. Where does he go? We must first analytically extend the Reissner-Nordstrom metric to cover the whole manifold, i.e. we must make the manifold *geodesically complete.* An analytic representation of the maximally extended spacetime can be obtained by replacing r and t by u and v defined by

$$\tan u = -e^{-\alpha(t-r_*)}, \tag{3.5}$$

$$\tan v = e^{\alpha(t+r_*)}, \tag{3.6}$$

where $\alpha = \frac{r_+ - r_-}{4r_+^2}$ and $r_* = r + \frac{r_+^2}{r_+ - r_-}\ln(r - r_+) - \frac{r_-^2}{r_+ - r_-}\ln(r - r_-)$.

In terms of these coordinates the Reissner-Nordstrom line element is

$$c^2 d\tau^2 = -\frac{4}{\alpha^2}\left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right)\csc 2u \csc 2v du dv - r^2 d\theta^2 - r^2 \sin^2\theta \, d\phi^2, \tag{3.7}$$

where r is defined implicitly by

$$\tan u \tan v = -e^{2\alpha r}(r - r_+)^{\frac{1}{2}}(r - r_-)^{-\frac{r_-^2}{2r_+^2}}. \tag{3.8}$$

In figure 3.2 below, ingoing and outgoing null geodesics are at 45°. This enables

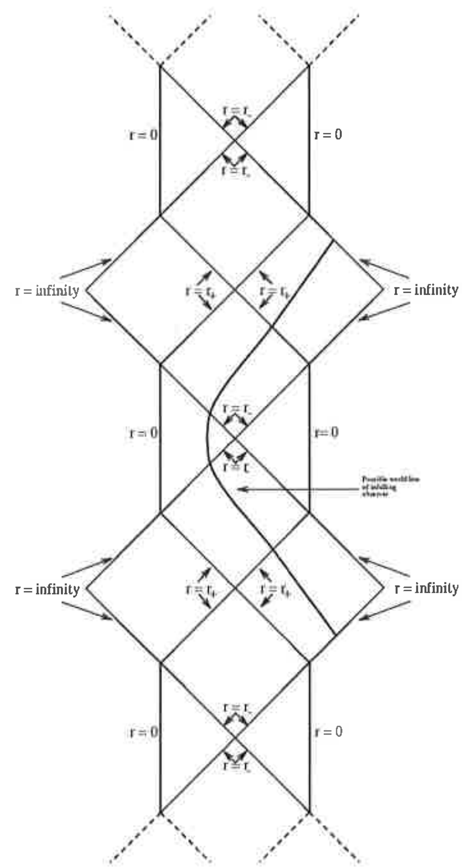

Figure 3.2: The Analytic Extension of Reissner-Nordstrom Spacetime

us to examine pictorially the underlying spacetime. If we follow the path of our explorer as he crosses $r = r_-$, he misses the singularity at $r = 0$, crosses $r = r_-$ again and re-emerges into another asymptotically flat spacetime. It seems the the Reissner-Nordstrom black hole can act as a bridge or *wormhole* to another universe. The collapsing star, or whatever matter falls in after it, falls on through the wormhole and out into, presumably, a cosmos much like our own.

3.2 The Kerr Solution

The Kerr solution [Ker63] is an exact solution which represents the stationary axisymmetric asymptotically flat spacetime outside a rotating body. In *Boyer and Lindquist coordinates* the Kerr metric can be given as

$$ds^2 = -\rho^2\left(\frac{dr^2}{\Delta} + d\theta^2\right) - (r^2+a^2)\sin^2\theta\, d\phi^2 + dt^2 - \frac{2mr}{\rho^2}(a^2\sin^2\theta\, d\phi - dt)^2, \quad (3.9)$$

where $\rho^2(r,\theta) = r^2 + a^2\cos^2\theta$ and $\Delta(r) = r^2 - 2mr + a^2$. The constants m and a represent the mass and angular momentum per unit mass of the rotating body.

The Kerr solution reduces to the Schwarzschild solution when $a = 0$. The line element has apparent singularities at $\rho^2 = 0$ and $\Delta = 0$. The solution of $\Delta = 0$ is given by

$$r_\pm = m \pm \sqrt{m^2 - a^2}, \quad (3.10)$$

and thus we have to consider two different situations:
(i) If $a^2 > m^2$ the metric is nonsingular everywhere except for the irremovable singularity at $r = 0, \theta = \frac{\pi}{2}$. In this case we also have a naked singularity as we again have no event horizon to clothe it.
(ii) If $a^2 \leq m^2$ the metric has additional singularities at r_+ and r_-. They will turn out to be coordinate singularities that will correspond to an event horizon and a Cauchy horizon respectively, just as in the Reissner-Nordstrom solution.

Now $\rho^2 = 0$ if and only if $r = 0$ and $\theta = \frac{\pi}{2}$ (the equatorial plane). To increase insight into this singularity, let us transform to *Kerr-Schild coordinates* $\left(x, y, z, \hat{t}\right)$, where

$$x + iy \;=\; (r + ia)\sin\theta\, e^{i\int(d\phi + \frac{a}{\Delta}dr)}, \quad (3.11)$$

$$z = r\cos\theta, \tag{3.12}$$

$$\hat{t} = \int \left(dt + \frac{r^2 + a^2}{\Delta} dr \right) - r. \tag{3.13}$$

In terms of these coordinates the Kerr metric is

$$\begin{aligned} ds^2 &= d\hat{t}^2 - dx^2 - dy^2 - dz^2 - \\ & \frac{2mr^3}{r^4 + a^2 z^2} \left[\frac{r(xdx + ydy) - a(xdy - ydx)}{r^2 + a^2} + \frac{z}{r} dz + d\hat{t} \right]^2, \end{aligned}$$

where r is defined implicitly by

$$r^4 - r^2(x^2 + y^2 + z^2 - a^2) - a^2 z^2 = 0. \tag{3.14}$$

We can see from equation (3.14) that surfaces of constant r correspond to confocal ellipsoids with principle axes coinciding with the coordinate axes. These ellipsoids degenerate for $r = 0$ to the disc $x^2 + y^2 \leq a^2, z = 0$, and thus the point $(r = 0, \theta = \frac{\pi}{2})$ corresponds to the ring $x^2 + y^2 = a^2, z = 0$. Therefore the Kerr solution is singular on a ring and not at a point as were the Schwarzschild and Reissner-Nordstrom solutions.

For the following discussion assume $0 < a^2 < m^2$. *Frame dragging* is an interesting property of the Kerr black hole. An observer is said to be *stationary* relative to the local geometry if and only if he moves along a world line of constant (r, θ). An observer is said to be *static* relative to the black hole's asymptotic Lorentz frame if and only if he moves along a world line of constant (r, θ, ϕ). The four-velocity of a stationary observer in Boyer and Lindquist coordinates is given by

$$\mathbf{u} = u^t \left(\frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \phi} \right), \tag{3.15}$$

and thus a stationary observer is static if and only if $\Omega \equiv \frac{d\phi}{dt}$ is zero.

The angular velocity Ω of a stationary observer is limited because his four-velocity $\mathbf{u}$ must lie within his future light cone and thus we have

$$\Omega_{\min} < \Omega < \Omega_{\max}, \tag{3.16}$$

where

$$\Omega_{\min} = \omega - \sqrt{\omega^2 - \frac{g_{tt}}{g_{\phi\phi}}}, \quad (3.17)$$

$$\Omega_{\max} = \omega + \sqrt{\omega^2 - \frac{g_{tt}}{g_{\phi\phi}}}, \quad (3.18)$$

$$\omega \equiv \frac{1}{2}(\Omega_{\min} + \Omega_{\max}) = \frac{2mra}{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta}. \quad (3.19)$$

We note that far from the black hole, we have $r\Omega_{\min} = -1$ and $r\Omega_{\max} = +1$. This corresponds to the standard limits imposed by light in flat spacetime. We can

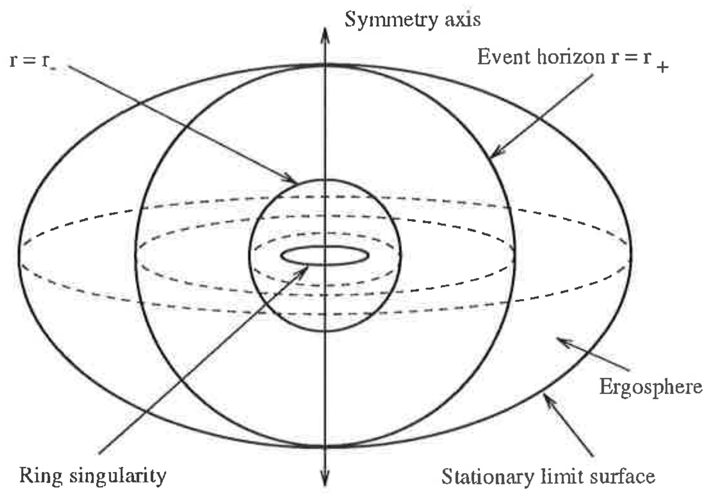

Figure 3.3: The Ergosphere

also see that $\Omega_{\min}$ increases as r decreases and by the time we reach

$$r = r_s(\theta) = m + \sqrt{m^2 - a^2\cos^2\theta}, \quad (3.20)$$

$\Omega_{\min}$ becomes zero. This means that all observers inside $r = r_s(\theta)$, no matter how hard they blast their rockets, can never have zero angular velocity relative to distant stars. For this reason $r = r_s(\theta)$ is called the *static limit* and the region $r_+ < r < r_s(\theta)$ is called the *ergosphere,* see figure 3.3 above.

Like the Schwarzschild and Reissner-Nordstrom solutions, we can analytically extend the Kerr metric to cover the whole spacetime manifold. As was shown, the $r = 0$ singularity corresponds to a ring and we can actually analytically continue r from positive values through the disc $x^2 + y^2 < a^2, z = 0$ to negative values.

We could not do this for the Schwarzschild and Reissner-Nordstrom solutions. As usual we can also analytically extend our metric through the event horizon $r = r_+$ and the Cauchy horizon $r = r_-$, see [HE73].

Figure 3.4 below is a pictorial representation of the maximally extended Kerr solution along the axis of symmetry. The dotted lines correspond to lines of constant r and the regions I, II and III represent $r_+ < r < +\infty$, $r_- < r < r_+$ and $-\infty < r < r_-$ respectively. Just as the Reissner-Nordstrom solution is a charged version of the Schwarzschild solution, the *Kerr-Newman solution* is a charged version of the Kerr solution. It is of the same form as equation (3.9) where $\Delta = r^2 - 2mr + a^2$ is replaced by $\Delta = r^2 - 2mr + a^2 + e^2$. It can be shown that the global properties of the Kerr-Newman solution are very similar to those of the Kerr solution [MTW73].

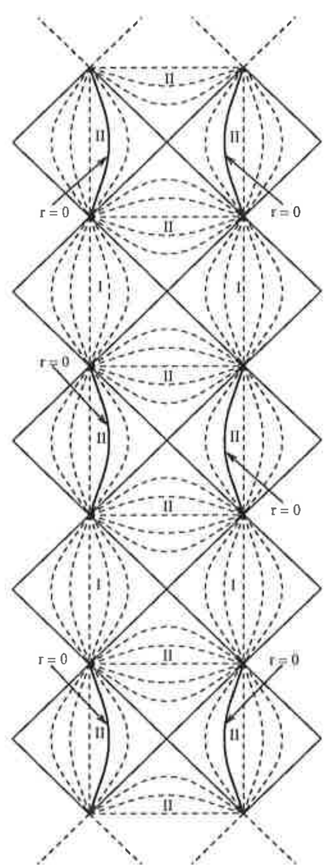

Figure 3.4: The Analytic Extension of Kerr Spacetime

CHAPTER IV

Spacetime Singularities

"Nature and Nature's laws lay hid in night.
God said, Let Newton be! and all was light."

Alexander Pope, 1688-1744

"It did not last, the Devil howling,
Ho let Einstein be, restored the status quo."

Sir John Squire, 1884-1958

4.1 Asymmetrical Gravitational Collapse

Symmetric gravitational collapse produces black holes with spacetime singularities, but are these singularities a pathology of the high symmetry involved or will they arise in general asymmetric collapse? To answer this question, we will first need to get familiar with a few new concepts.

Definition [Pen65].
A *closed trapped surface* is defined to be a C^2 closed, spacelike, two-sphere T_2 with the property that the two systems of null geodesics which meet T_2 orthogonally converge locally in future directions at T_2.

In figure 4.1, a pictorial representation of a closed trapped surface is given. The surfaces S_1 and S_2 are the sufaces formed by light beams emitted inwards and outwards from the surface T_2 respectively. Notice the surface T_2 need not be symmetric in any way, and the light beams do not need to focus at an exact point. But they do cross each other in a complicated way to form continuous surfaces, with no holes.

We can see that closed trapped surfaces will form in any region of spacetime where gravity is so strong that outward directed light gets dragged inwards. Thus for black holes, closed trapped surfaces will definitely form within event horizons.

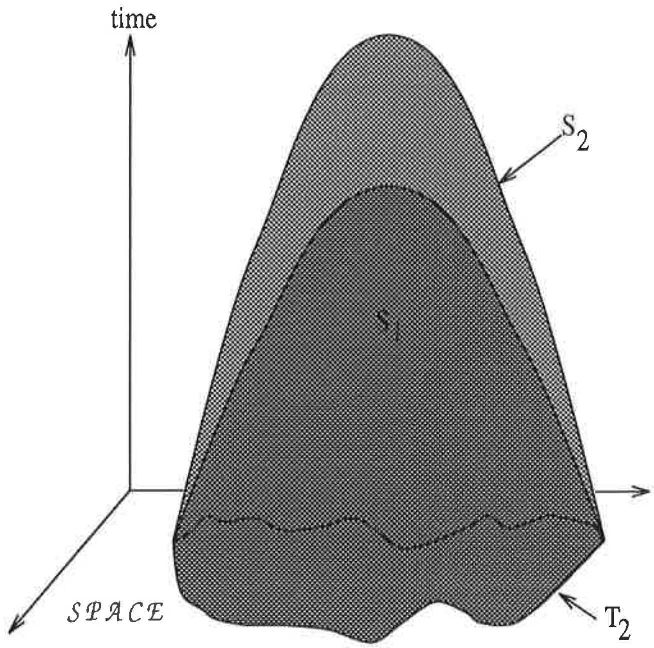

Figure 4.1: A Pictorial Representation of a Closed Trapped Surface

In fact, if there is a sufficient concentration of matter in a small enough region, a closed trapped surface will form [Cla87]. We proceed by giving a precise definition of singularities [Sch71].

Definition

If a spacetime (M, g) is timelike or null geodesically incomplete, we say that the termination point of a timelike or null incomplete curve, together with all adjacent termination points, is a *singularity.*

Notice that the definition makes no mention of infinite curvature. When using this more general definition, a singularity is really a hole or edge of spacetime and need not be a point where we have infinite spacetime curvature. A black hole produced by an infinitely compacted star would still have a singularity at $r = 0$ because timelike and null geodesics terminate there. The usual idea of what singularities are, is contained within the definition. Singularities which also have associated infinte curvature are generally known as *curvature singularities.*

I will now state one of the most powerful of a wide class of singularity theorems. For a proof see [HE73].

Theorem [HP70]

A spacetime (M, g) necessarily contains incomplete, inextendable timelike or null geodesics if, in addition to Einstein's equations the following four conditions hold;

(i) $R_{ab}K^aK^b \geq 0$ for every non-spacelike vector $\mathbf{K}$ (reasonable energy condition);

(ii) The manifold is "general", that is every non-spacelike geodesic contains a point at which $K_{[a}R_{b]cd[e}K_{f]}K^cK^d \neq 0$, where $\mathbf{K}$ is the tangent vector to the geodesic.

(iii) There are no closed timelike curves (causality condition);

(iv) The manifold contains a closed trapped surface.

From the theorem we can see that if certain reasonable assumptions are made, deviation from spherical symmetry cannot prevent spacetime singularities from arising.

4.2 Naked Singularities

The theorem of the previous section proved that under certain reasonable assumptions, a trapped surface is a sufficient condition to produce a singularity. It is not a necessary condition and the question remains, can singularities form in the absence of closed trapped surfaces? If they can, they will be exposed for the whole universe to see. In 1969 Roger Penrose proposed what he called a *cosmic censor hypothesis* : [Pen69]

> "... We are thus presented with what is the perhaps the most fundamental unanswered question of general-relativistic collapse theory, namely: does there exist a 'cosmic censor' who forbids the appearance of naked singularities, clothing each one in an absolute event horizon?"

This is now known as the *weak cosmic censor hypothesis.* A stronger idea is the *strong cosmic censor hypothesis,* see [Pen74], which asserts that a "physically reasonable" spacetime must be globally hyperbolic.

We know that the singularity in the Schwarzschild black hole is not visible to us. Even inside the event horizon the singularity always lies in our future, until we hit it. In contrast, for the Reissner-Nordstrom and Kerr black holes, once we travel inside the inner Cauchy horizon $r = r_-$, the singularity lies in our past as well as our future and is thus naked. Is this a counterexample to the cosmic censor hypothesis? Examination of the time dislocation of an infalling observer as he crosses $r = r_-$, reveals that he can witness, by looking outwards, the whole future history of the universe passing by in one fleeting moment. Besides vapourizing him, the energy of all the light coming in will create another singularity along the Cauchy horizon $r = r_-$. Thus we cannot even get inside $r = r_-$ and the singularity at $r = 0$ is indeed not naked.

As mentioned earlier, if the charge or angular momentum per unit mass of a black hole is large enough, no event horizon forms and its singularity becomes naked. It can be shown that this also is a false assumption [Dav81].

Another argument for violations of strong cosmic censorship comes from what is known as *Hawking radiation* [Haw75]. By considering quantum processes around black holes, Hawking discovered that black holes actually evaporate. From this evaporation process, it can be shown that if the universe is open or flat and given enough time, a contradiction with global hyperbolicity can be obtained. This argument, it should be noted, does not itself show there exists a naked singularity in the sense of an incomplete spacelike or null geodesic. It is also internally inconsistent. One of the physically reasonable conditions for cosmic censorship is an energy condition which is violated when the Hawking quantum processes are taken into account.

On the other hand, several theorems have been proved. While not establishing cosmic censorship, they have limited the sorts of naked singularities to the kind that can occur where the curvature is fairly low [New85].

This was the original idea of cosmic censorship in the beginning. Singularities should be associated with curvature, and that this curvature in turn should focus geodesics that lead to closed trapped surfaces that clothe the singularities. Thus it seems that if there is a cosmic censor, he does his job with great subtlety.

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